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EPL, 78 (2007) 65001

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Wave-particle interactions in collisionless plasmas: The failure of Vlasov approximation in describing the approach to statistical equilibrium

V. Carbone, R. De Marco, F. Valentini and P. Veltri

Dipartimento di Fisica, Università della Calabria, and CNISM, Sezione di Cosenza - 87036 Rende (CS), Italy

received 26 February 2007; accepted in final form 3 May 2007
published online 25 May 2007

PACS 52.20.-j – Elementary processes in plasmas
PACS 05.45.-a – Nonlinear dynamics and chaos
PACS 52.65.Rr – Particle-in-cell method

Abstract – We investigate the dynamical approach towards equilibrium during the wave-particle interaction process in plasmas. Particle in Cell (PIC) simulations are compared with a Fermi-like model where particle trapping is described as elastic collisions of particles with two barriers whose amplitude is proportional to the energy of the wave. The system approaches statistical equilibrium through a phase mixing described by a filamentation of the phase space, a dynamical system analogous of the Gibbs process of laminar mixing. The system of trapped particles evolves towards the state of maximum entropy and wave-particle interaction creates correlations in the phase space particle trajectories, making Vlasov equation fail in describing the system dynamics.

Wave-particle interaction in plasmas is a basic physical process because it is responsible of wave dissipation even in the absence of collisions [1]. The phenomenology of the linear process, introduced originally by Landau [2], is quite intuitive: while the bulk of particles is not affected by the interaction, a particle travelling at speed \( v \) near the wave phase velocity \( v_p \), gains energy if its velocity is just below \( v_p \) and loses energy just above \( v_p \). In particular, since for a Maxwellian distribution of particle velocities, one has \( \frac{\partial f}{\partial v} \big|_{v=v_p} < 0 \), the wave must be exponentially damped. As conjectured by O’Neil [3], the damping can be prevented in the nonlinear regime, due to the trapping of particles that oscillate in the wave potential well [1].

Even if the situation is not so clear, it can be realized that trapped particles both can gain and lose energy thus generating a saturation of damping. The problem has been investigated numerically [4–8] by solving the Vlasov–Poisson system, and only recently it has been realized that the framework of dynamical systems can be useful to investigate the behaviour of particles in the wave field [7,9–11]. Numerical and analytical results show the existence of a critical initial amplitude of the wave that marks the transition between the Landau scenario (for low initial amplitudes) and a scenario where the damping is prevented (high initial amplitudes) [12]. The latter seems to be characterized by a series of oscillations of the wave amplitude around a constant saturation value.

Recently [8] the physical meaning of these oscillations have been questioned, on the basis of the fact that a truly dissipationless simulation of the Vlasov-Poisson system cannot be made. As a consequence, it is not clear how the system approaches to the statistical equilibrium in dissipationless plasmas. In previous works [4–8], wave-particle interaction has been studied almost exclusively using Vlasov-Maxwell simulations. Nevertheless, in the present paper, in order to analyze the long-time behaviour of the wave-particle interaction in nonlinear damping regime and to reproduce the approach to a statistical equilibrium of the system, we use particle simulations that in some way retain the intrinsic discreteness of the system, missed by the continuous Vlasov description. Standard PIC numerical simulations [13,14] performed in the regime of nonlinear Landau damping are compared with the numerical results of a Fermi-like model [11].

PIC simulations follow the electron dynamics in the \( x \)-direction, which is the direction of wave propagation (the ions are considered motionless). The equations of motion of a large number of macro-particles (\( N = 10^7 \)) are integrated numerically through a standard second-order leap-frog scheme. For convenience, we scale time by the inverse plasma frequency \( \omega_p^{-1} \), where \( \omega_p = \sqrt{4\pi ne^2/m} \).
and \( n \) is the electron density. Length is scaled by the Debye length \( \lambda_D = v_{th}/\omega_p \). With these choices, velocity is scaled by the electron thermal velocity \( \lambda_D \omega_p = v_{th} \) and electric field by \( \sqrt{4\pi n e^2 v_{th}^3} \). The electron phase space domain for the simulation is \( D = [0, L_x] \times [-v_{max}, v_{max}] \), where \( v_{max} = 5 \). Periodic boundary conditions in physical space are imposed, then Poisson’s equation is solved using a standard Fast Fourier Transform (FFT) routine.

The initial distribution function is a Maxwellian in the velocity space, over which a modulation in the physical space with amplitude \( A \) and wave vector \( k = 2\pi/L_x \) is superposed:

\[
f(x, v, t = 0) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} [1 + A \cos(kx)].
\]

The length of the spatial domain is chosen to be \( L_x = 20 \), the amplitude of the perturbation is \( A \approx 0.064 \). The simulations follow the evolution of electrons for many plasma periods \( (t_{max} = 4000) \). For a PIC simulation with one spatial dimension and with grid spacing smaller than the Debye length, the effective collision time (see [13]) is longer than \( t = (1/\omega_p)n\lambda_D \), where \( n = N/L_x \) is the \( 1-D \) density of particles in the simulation. In scaled units, the collision time is larger than \( t = N/L_x \approx 10^6 \) (for \( L_x = 20\lambda_D \)), which is much longer than duration of our runs \( t_{max} = 4000 \). In these conditions, our simulations can be considered fully collisionless.

In the Fermi-like model the wave-particle interaction is described as the result of a large number of collisions between an ensemble of particles and two infinitely massive, one-dimensional barriers with variable length, set at a fixed distance \( L = 2 \). The whole system moves with speed \( v_p \), so, in this frame of reference, each elastic collision of a particle with a barrier will reverse the direction of motion of the particle, \( i.e., v' = -v \) (\( v' \) is the velocity after the collision). After each wave-particle collision, the amplitude of the barrier \( A \) varies according to \( A(t') = [A(t)]^2 + \sigma(t)\Delta E 1/2 \), where \( \sigma(t) = v_{wp}/(|v_{wp}|) \) and \( \Delta E = 4v_{wp} \). Finally, the energy of the wave \( E_W \) is proportional to the square amplitude of the barrier \( (E_W \approx |A(t)|^2) \).

In the Monte Carlo simulation of the above model, velocities are normalized to the thermal velocity of plasma, \( i.e., u = v/v_{th} \) and \( u_p = v_p/v_{th} \). Each particle is identified by the random initial position \( x_j(0) \) (\( j = 1, 2, \ldots, N \), being \( N \) the number of particles) and by a constant speed \( u_j \), extracted from the normalized distribution function:

\[
f_0(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}(1 - uu_p).
\]

The numerical domain is given by \( D_L = [-L/2, L/2] \times [-\sqrt{A_0}, \sqrt{A_0}] \) with \( A_0 = A(0) = 1 \). Particles are placed between the barriers at different altitudes \( y_j \leq A_0 \), taken equal to their square velocity \( y_j = u_j^2 \). The dynamics of the \( j \)-th particle between two collisions is trivially described by the equation of motion of a point mass moving at constant speed \( u_j \). For our simulations, we choose \( u_p = 1 \).

![Fig. 1: At the top: Time evolution of the electric-field amplitude (PIC simulation). At the bottom: Time evolution of the barrier amplitude (Fermi-like model).](image)

The top plot in fig. 1 shows the numerical results of the PIC code for the time evolution of the amplitude of the electric-field spectral component \( E_k(t) \), while the bottom plot reports the typical time evolution of the barrier amplitude resulting from the Fermi-like model. As can be seen in both figures, after an initial decay, the amplitude starts oscillating around a saturation value. The amplitude of these oscillations around an almost constant value initially decreases, then oscillates regularly and finally, for long times, the dynamics becomes completely stochastic. Even if discrete-particle effects in the nonlinear evolution of wave-particle interaction have been discussed in the past [15], the long-time behaviour, that resembles the O’Neil scenario for the nonlinear Landau damping [3], where the oscillations in the electric envelope disappear in the long time limit, has never been obtained in numerical simulations, even if oscillations with decreasing amplitude have been observed [5–8].

The time evolution described in fig. 1 is determined by the interaction of the particles with the potential well of the wave, that is sinusoidal for the PIC simulation, and squared for the Fermi-like model. The particle dynamics is similar in both cases: particles whose energy is larger than the potential barrier free stream unperturbed; a few particles at the border of the trapped region can be trapped and detrapped periodically, performing more or less long flights [7]; finally, the bulk of particles with energy well below the potential barrier is ever trapped.

Even if the time behaviour of the bulk of trapped particles is trivial, their dynamical role is extremely important. Each of these particles follows a closed path in
Fig. 2: PIC simulation: phase space scatter plot at four different times (left panels); distribution function \( f(x_M, u) \) as a function of \( u \), at four different times (right panels).

the phase space \((x, u)\); this path has an elliptical form for the PIC simulation and a rectangular one for the Fermi-like model. Each path represents a torus of the system, and particles move around each closed path at different times, according to their speed. In the left panels in fig. 2, we report the phase space \((x, u)\) for the PIC code (in the region around the wave phase velocity \( v_p \approx 3.7 \)) while the phase space for the Fermi-like model is shown in the left panels in fig. 3. Each point in the scatter plots represents the position \( x_j \) and the velocity \( u_j \) of the \( j \)-th particle at time \( t \).

The particle distribution function can be defined through the usual coarse graining of a point system in statistical mechanics, say \( f(x, u, t) \Delta x \Delta u \) represents the number of points that, at time \( t \), lie within the box of (narrow but finite) amplitude \( \Delta = \Delta x \Delta u \). From fig. 2 and 3 we can see that, as the time increases, the dynamics of particles generates a filamentation of the structure in phase space. Filaments become narrower, up to a time where an apparently complete mixing settles up. In the right panels in fig. 2, the distribution function \( f(x_M, u, t) \) defined above is plotted as a function of the velocity \( u \) for a given point in the physical space \( x_M \) at different times, for the PIC simulation. The point \( x_M \) corresponds to the coordinate in the physical space where the velocity width of the trapped region is maximum. In fig. 3 (right panels), the electron distribution \( f(u, t) \) (averaged on \( x \)) is shown at different times, for the Fermi-like model. From these two figures, we argue that the process of filamentation of phase space corresponds to the presence of “stripes”. As time increases stripes become narrower and, for long times, the distribution gets flat.

A quick reflection is enough to realize that the occurrence of velocity stripes produces a synchronization of collisions on each barrier. As the stripes get narrower, less and less particles collide almost synchronously on a given side of the potential well. Actually [7] this is the dynamical process that is responsible for the presence of the amplitude oscillations shown in fig. 1. In fact when the stripes are large enough, the synchronization of collisions on a given barrier is able to sustain or decrease the amplitude of the oscillations around the saturation value. When the filamentation goes on producing narrower velocity stripes, the synchronization of collisions is felt by a lower number of particles and oscillations become less regular up to a complete phase mixing. This means that, for long times, the system has reached the equilibrium configuration, and the approach to equilibrium is described by a filamentation of the phase space. The approach of both our conservative particle systems to statistical equilibrium through a laminar filamentation is analogous to the well-known phenomenon of laminar mixing between two fluids in statistical

65001-p3
Fig. 3: Fermi-like model: phase space scatter plot at four different times (left panels); distribution function $f(u)$ (averaged on $x$) at four different times (right panels).

mechanics that has been investigated by Gibbs. In analogy to the bakers’ transformation [16,17], the Fermi-like dynamical system plays the role of an area-preserving two-dimensional map for the wave-particle interaction process. This process of laminar filamentation saturates when the width of the phase space filaments becomes of the same order of or smaller than the characteristic scale of discreteness of the system (inter-particle separation). Structures whose width is smaller than this characteristic length are no more distinguishable (Gibbs laminar-mixing process). In Vlasov theory, no characteristic length is present, meaning that in principle a Vlasov system could keep producing narrower and narrower filaments and phase-space structures with increasing time, thus never reaching the mixing stage, where the filaments are no longer resolved. In conclusion, the Gibbs mixing phenomenon is naturally ruled out from Vlasov theory.

This approach to equilibrium can be even better clarified with a simple argument: in the absence of Coulomb collisions between particles, the trapped region can be thought of as an isolated system with respect to the free region, where particles do not exchange energy with the wave (except for a few particles that can escape from the well and be retrapped after a finite time). It is straightforward to show that by imposing statistical entropy of the system of trapped particles $S = -\int D_x f \log f dx du$ to be maximum, while conserving the total number of particles and assuming an infinite energy reservoir (the wave energy), the distribution function turns out to be flat over the velocity interval. As a consequence, the system naturally evolves towards the maximum entropy state, through the laminar mixing process discussed above. In fig. 4, we show the time evolution of the entropy variation $\Delta S = S(t) - S(0)$, for the Fermi-like model, with three different numbers of numerical cells in phase space $N_x$. After the initial growth, $\Delta S$ reaches a saturation level. The jump in the entropy of the system corresponds to the entropy difference $\Delta S^{th}$ between a flat velocity distribution (see the bottom right plot in fig. 3) and a decreasing function (see the top right plot in fig. 3). The quantity $\Delta S^{th}$ is indicated in the figure by a horizontal solid line. It is worth noting that the jump in $\Delta S$ does not depend on the numerical resolution.

The curves in fig. 4 can be fitted with a function $\Delta S \propto 1 - \exp(-t/\tau_G)$, $\tau_G$ being the time when a complete Gibbs mixing is reached. Repeating such simulations for many different numbers of particles $N$, gives the scaling law $\tau_G \propto N^\alpha$, where $\alpha = 0.55 \pm 0.02$.

As a last consideration, we analyze the wave-particle interaction and the trapping phenomenon from the point of view of the statistical mechanics. In collisionless systems, the dynamics of particles trapped in the wave field is usually described through the Vlasov equation; this equation is generally viewed as the limit of the
Boltzmann equation, when particle correlations are negligible. In wave-particle interaction, when trapped dynamics takes place, long-range correlations between particles are created by the wave potential that traps the particles and connects their trajectories in phase space. To substantiate this conjecture, we numerically evaluate the two-particle correlation function through the PIC code and the Fermi-like model and finally compare the results.

The normalized two-particle correlation function is

\[ C_2(i, j) = \frac{\sum_{g=1}^{n_c} g(i, i') g(j, j')} {n_c^2}, \]

where \( f_2 \) and \( f_1 \) represent the two-particle and the single-particle coarse-grained distributions, respectively; the integers \( i \) and \( j \) indicate the \( i \)-th and \( j \)-th cells of area \( \Delta \) in phase space. The distributions \( f_2 \) and \( f_1 \) can be evaluated dividing each phase space cell of area \( \Delta \) in \( n_c \) subcells of area \( \Delta' \). Using this subgrid in phase space, we define

\[ f_2(i, j) = \frac{\sum_{i'=1}^{n_c} g(i, i') g(j, j')} {n_c}, \quad f_1(i) = \frac{\sum_{i'=1}^{n_c} g(i, i')} {n_c}, \]

where the integer \( i' \) refers to the \( i' \)-th subcell contained in a generic cell in phase space and \( g(i, i') \) is simply the number of particles in a subcell of area \( \Delta' \), divided by the total number of particles \( N \) (fine grained distribution).

In fig. 5, we show in logarithmic scale the time evolution of the maximum value of the two-particle correlation function \( C_2 \) in phase space, from the PIC code (top plot) and from the Fermi-like model (bottom plot). In both cases, the evolution of \( \max(C_2) \) is followed up to the time when the first two oscillations occur in the signals described in fig. 1. In the first trapping period, when the wave starts trapping the resonant particles, thus creating correlations in their phase space trajectories, we observe an increasing of the two-particle correlation function, that finally saturates when regular oscillations occur in fig. 1. This shows that wave-particle interaction creates correlation in the phase space particle trajectories.

An analogous increasing in the correlation function has been discussed in ref. [18], where the authors numerically integrated the equations of motion of a certain number of test-particles in the electric field produced through an eulerian Vlasov simulation.

The fact that both the entropy of the system and the correlation function increase when trapping becomes important suggests that Vlasov description fails in describing wave-particle interaction. In fact, in Vlasov theory the feedback of correlation growth on the dynamical evolution of the particle distribution function, which is described by the collision term, is neglected. Moreover Vlasov equation cannot describe increasing entropy processes, because as time increases it produces continuous structures at smaller and smaller scales, missing the intrinsic discreteness of the plasma system. In a recent paper by Galeotti and Califano [8], the authors pointed out other important limitations of the Eulerian approach in the analysis of the long-time evolution of a Vlasov-Poisson plasma, due to numerical diffusion effects.

In conclusion, in this paper we show for the first time that the long-time behaviour of nonlinear Landau damping is analogous to the phase mixing process described by O’Neil [3]. The phase mixing is described by the filamentation of the phase space, a dynamical system analogous of the Gibbs process of laminar mixing described by the bakers’ map [16]. The Gibbs mixing time \( \tau_G \) increases with the number of simulation particles as \( \tau_G \propto N^{0.55} \). As we discussed above, in principle this laminar mixing process cannot be described within the Vlasov theory. The system of trapped particles evolves towards the state of maximum entropy and wave-particle interaction creates correlations.
in the phase space particle trajectories, making Vlasov equation fail in describing the system dynamics.

Finally, we would like to point out that the Fermi-like model plays a pedagogically fundamental role in the analysis of wave-particle interaction phenomena. Even though the description of the trapping interaction is not as complete as that obtained through PIC simulations (or Vlasov simulations), the Fermi model catches the fundamental physics of the process.

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