Energetic particle transport in the presence of magnetic turbulence: influence of spectral extension and intermittency

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ABSTRACT

The transport of energetic particles in the presence of magnetic turbulence is an important but unsolved problem of space physics and astrophysics. Here, we aim at advancing the understanding of energetic particle transport by means of a new numerical model of synthetic magnetic turbulence. The model builds up a turbulent magnetic field as a superposition of space-localized fluctuations at different spatial scales. The resulting spectrum is isotropic with an adjustable spectral index. The model allows us to reproduce a spectrum broader than four decades, and to regulate the level of intermittency through a technique based on the $p$-model. Adjusting the simulation parameters close to solar wind conditions at 1 au, we inject $\sim 1$ MeV protons in the turbulence realization and compute the parallel and perpendicular diffusion coefficients as a function of spectral extension, turbulence level, and intermittency. While a number of previous results are recovered in the appropriate limits, including anomalous transport regimes for low turbulence levels, we find that long spectral extensions tend to reduce the diffusion coefficients. Furthermore, we find for the first time that intermittency has an influence on parallel transport but not on perpendicular transport, with the parallel diffusion coefficient increasing with the level of intermittency. We also obtain the distribution of particle inversion times for parallel velocity, a power law for more than one decade, and compare it with the pitch angle scattering times observed in the solar wind. This parametric study can be useful to interpret particle propagation properties in astrophysical systems.

Key words: diffusion – magnetic fields – methods: numerical – solar wind.

1 INTRODUCTION

The transport of energetic particles in the heliosphere and in the interstellar medium is a crucial and open problem in astrophysics. Particle transport is strictly related to the properties of magnetic field turbulence. For instance, magnetic turbulence gives rise to particle pitch angle diffusion, which influences transport parallel to the background magnetic field; also, magnetic turbulence gives rise to transverse displacements because of drift motions in inhomogeneous fields and to the random walk of magnetic field lines, which induce cross field transport of particles. Understanding transport is relevant for electron acceleration in solar coronal loops (Galloway, Helander & MacKinnon 2006; Zharkova et al. 2011), for the prediction of the arrival of solar energetic particles in the vicinity of the Earth (Dalla et al. 2003; Luhmann et al. 2010), and for assessing particle propagation in the solar wind, where regimes ranging from scatter-free (or ballistic) to diffusive (determined by pitch angle diffusion) are reported (Lin 1974; Zhang et al. 2003; McKibben 2005; Perri & Zimbardo 2007, 2009). The acceleration of galactic cosmic rays at supernova remnant shocks depends in a sensitive way on the particle diffusion coefficient and on the turbulence level (Drury 1983; Bell 2004; Amato & Blasi 2006, 2009; Amato 2014). Recently, the influence on the acceleration processes at shocks of non-diffusive, non-Gaussian transport regimes, which encompass a non-linear growth of the mean square displacement with time, has also been considered (Duffy et al. 1995; Perri & Zimbardo 2012a; Zimbardo & Perri 2013; Lazarian & Yan 2014).

Many studies have addressed the properties of energetic particle transport, both from a theoretical (Giacalone & Jokipii 1999; Teufel & Schlickeiser 2002; Matthaeus et al. 2003; Shalchi, Bieber & Matthaeus 2004; Shalchi 2015) and from an observational point of view (Reames 1999; Mazur et al. 2000, and many others). However, also due to non-linear effects, kinetic effects, and complex multiscale structures, a complete understanding is still lacking. Consequently, a large number of numerical studies have been devoted to the calculation of particle diffusion coefficients for transport parallel and perpendicular to the average magnetic field. The
obtained transport regimes vary widely, see the review by Zimbardo et al. (2012). It has been found that several parameters can influence the transport of energetic particles, like the turbulence level $\delta B / B_0$, the anisotropy of turbulence, and the ratio between the particle Larmor radius $R_i$ and the turbulence correlation length $l$. In particular, Giacalone & Jokipii (1999) have studied particle transport with a numerical model in which turbulence can be either isotropic or composite (slab plus 2D with a prevalence of modes perpendicular to the average magnetic field). They have found that transport is normal both parallel and perpendicular to the average magnetic field $B_0$, that the perpendicular diffusion coefficient grows with the turbulence level and with the particle energy, and that the ratio between perpendicular and parallel diffusion coefficients $D_\perp / D_\parallel$ has values of the order of 0.02–0.04, even for strong turbulence levels $\delta B / B_0 = 1$. Casse, Lemoine & Pelletier (2001) have performed a similar study additionally considering $\delta B / B_0 > 1$, which is relevant for galactic cosmic rays acceleration; they have found that in the limit $\delta B / B_0 \gg 1$ the ratio $D_\perp / D_\parallel \simeq 1$. However, both studies of Giacalone & Jokipii (1999) and Casse et al. (2001) use a turbulence model whose spectral extension is slightly more than two decades, while in the solar wind the extension of the magnetic spectrum of turbulence is often of the order of three–four decades (see e.g. Kiyani, Osman & Chapman 2015).

Anomalous transport corresponding to perpendicular subdiffusion was reported by Qin, Matthaeus & Bieber (2002a) in the case of slab turbulence, while parallel diffusion was found to be normal. The perpendicular subdiffusion is a result of particles tracing back the field lines after pitch angle diffusion, and is related to the so-called compound diffusion (Rechester & Rosenbluth 1978; Kôta & Jokipii 2000; Shalchi 2010; Tautz & Shalchi 2010; Shalchi 2015). This process is essentially non-Markovian (Kôta & Jokipii 2000), due to the memory effect induced by tracing backwards the (magnetostatic) field lines, which is possible in the case of low stochasticity of field lines. Indeed, for a composite model of slab plus 2D turbulence with 80 per cent of fluctuation energy in the 2D spectrum, Qin, Matthaeus & Bieber (2002b) have recovered diffusion also perpendicular to $B_0$, in agreement with the fact that the exponential separation of field lines is faster for 2D turbulence. Similar conclusions on subdiffusion for slab turbulence and recovery of diffusion for composite turbulence have been reached by Shalchi (2010) by means of an analytical modelling based on fourth-order correlations. More recently, Hussein & Shalchi (2016) used a composite slab plus 2D model to describe dynamical turbulence, characterized by an empirical time-dependent correlation function; this allows us to reproduce the diffusive mean free paths corresponding to the so-called Palmer consensus, but only assuming a fluctuation level $\delta B / B_0 = 0.5$ instead of 1. While the composite model of turbulence allows us to reproduce a very long spectrum, a shortcoming of this model is that only wavevectors perpendicular to $B_0$ and parallel to $B_0$ are present: oblique wavevectors are missing, and this can influence the wave-particle resonance condition.

The effect of turbulence anisotropy on particle transport has been studied by Zimbardo, Pommois & Veltri (2006) and Pommois, Zimbardo & Veltri (2007) with a fully 3D anisotropic numerical model, which includes wavevectors on all directions. It is found that for quasi-slab turbulence transport is anomalous, corresponding to parallel superdiffusion and to perpendicular subdiffusion. On the other hand, in the isotropic case parallel transport is superdiffusive, while perpendicular transport is normal. In the quasi-2D case, both parallel and perpendicular transport are found to be normal, confirming the results of Qin et al. (2002b). Anomalous, superdiffusive transport was also found with the composite turbulence model by Shalchi & Kourakis (2007). These results emphasize the importance of turbulence anisotropy. However, the spectral extension used by Zimbardo et al. (2006) and Pommois et al. (2007) was about one decade, much less than what is envisaged for space and astrophysical magnetohydrodynamic (MHD) turbulence.

Furthermore, in recent years it has been shown that solar-wind turbulence is highly intermittent (e.g. Marsch & Tu 1997; Sorriso-Valvo et al. 1999). Intermittency is one of the most typical properties of the turbulent energy cascade, consisting of the spatially non-uniform, non-Gaussian distribution of fluctuations, associated with highly localized and intense magnetic field variations, and usually revealed through the non-scalability of high-order structure functions (Veltri & Mangeney 1999). In this work, we present a new numerical realization of a 3D turbulent magnetic field based on the superposition of localized fluctuating magnetic fields at different spatial scales, giving origin to a power-law isotropic spectrum, and where, beside the other parameters, the level of intermittency can also be tuned. Intermittency is implemented in the numerical turbulence by means of a $p$-model, a well-known representation of intermittency (Meneveau & Sreenivasan 1987). The present turbulence model shares several aspects with other previous models (Juneja et al. 1994; Cametti, Carbone & Veltri 1998). However, it is based on a new algorithm which allows for a substantial reduction of computational costs (time and memory requirements) in 3D wide spectrum configurations. Consequently, it is possible to obtain a realistic, broad spectral extension at reasonable computational cost. Using these synthetic turbulent fields, we perform numerical simulations of particle transport in order to understand the influence of the turbulence spectrum extension, of intermittency, and of $\delta B / B_0$ on the transport regimes. We note that when the particle speed is substantially larger than the Alfvén speed (i.e. the typical speed at which MHD disturbances propagate), the magnetic fluctuations can be considered as static. In the same limit, the electric force is much smaller than the magnetic force and can be neglected, so that the so-called magnetostatic approximation is obtained. Such approximation is appropriate both for heliospheric energetic particles (for instance, $v \sim 5000$ km s$^{-1}$ for a 100 keV proton, while the Alfvén speed in the solar wind at 1 au is of the order of 50 km s$^{-1}$) and for galactic cosmic rays. Here, we choose simulation parameters corresponding to $\sim 1$ MeV protons in the solar wind at 1 au; this implies that $\delta B / B_0 = 0.5–1$, typically. The investigation of transport for larger turbulence levels, as appropriate for supernova remnant shocks (Amato & Blasi 2009; Amato 2014), is left for future work.

This article is organized as follows. In Section 2, we describe the structure and main properties of the numerical code; in Section 3, we present the numerical results as a function of the various parameters; and in Section 4, we give the conclusions.

## 2 SYNTHETIC TURBULENCE MODEL

As we have mentioned in previous section, a broad spectral width is an important ingredient for a realistic simulation of particle transport in turbulence. In our model, the largest scale is set equal to the typical correlation length in the solar wind turbulence at 1 au, $l_{\text{max}} \approx L \sim 5 \times 10^5$ km (e.g. Horbury et al. 1996). On the other hand, the energetic proton Larmor radius $R_i$ should also be resolved at small scales. A typical value for the solar wind conditions is $R_i = m_p c v / (eB) \sim 2 \times 10^2$ km. These two scales define the rigidity parameter $\rho = R_i / l_{\text{max}} \sim 4 \times 10^3$. If the model spectral width is defined as the largest to smallest scale ratio, $\xi = l_{\text{max}} / l_{\text{min}}$, where $l_{\text{min}}$ is the smallest scale considered in the spectrum, the condition $\xi > 1 / \rho$ must be satisfied to include all the relevant scales. With the
The turbulent magnetic field is modelled as a superposition of spatially localized eddies characterized by a fluctuating magnetic field, each eddy being associated with a cell. In order to improve statistical homogeneity, eddies at the largest scales $\ell_{\text{max}} < \ell \leq \ell_0$ are set to zero, where $\ell_{\text{max}}$ roughly corresponds to the correlation length. The turbulent magnetic field $B$ has the form:

$$\mathbf{B}(x, y, z) = B_0 + \sum_{m=0}^{N_c} \sum_{i,j,k=1}^{2^m} \delta \mathbf{B}^{(i,j,k,m)}(x, y, z),$$

where $B_0$ is a uniform background magnetic field, and $\delta \mathbf{B}^{(i,j,k,m)}(x, y, z)$ is the fluctuating magnetic field associated with the eddy $(i,j,k;m)$. This has the form:

$$\delta \mathbf{B}^{(i,j,k,m)} = a^{(i,j,k,m)} \nabla \times \mathbf{A}^{(i,j,k,m)},$$

where $a^{(i,j,k,m)}$ gives the fluctuating field amplitude while $\mathbf{A}^{(i,j,k,m)} \sim O(\ell_{\text{max}}/\ell_0)$ is a normalized vector potential. The index $m_{\text{min}}$ in equation (2) identifies the largest scale in the spectrum: $m_{\text{min}} = \log(\ell_0/\ell_{\text{max}})$. We used $\ell_{\text{max}}/\ell_0 = 1/4$. For a total number of scales $N_c = 16$ this gives a spectral width $\xi = \ell_{\text{max}}/\ell_{N_c} = 2^{14} \approx 1.6 \times 10^4$.

The fluctuating magnetic field $\delta \mathbf{B}^{(i,j,k,m)}$ is a three-component solenoidal vector field that is non-vanishing only inside a subdomain $\mathcal{V}^{(i,j,k,m)}$, which contains the corresponding cell $C^{(i,j,k,m)}$. The subdomain has the same spatial shape as the cell, but its size is twice the cell size, so neighbouring eddies partially overlap. Such a feature has the purpose of avoiding that $B$ vanishes at any surface border of adjacent cells, thus reducing the artificial spatial periodicity introduced by the regular cell lattice. The components of the vector potential $\mathbf{A}^{(i,j,k,m)}(x, y, z)$ are polynomial functions which have the property of vanishing at the border of the corresponding subdomain, along with all their spatial derivatives up to the fourth order. Thus, the total magnetic field $\mathbf{B}$ is a continuous function of $(x, y, z)$ along with its derivatives up to the third order. The spatial profile of $\mathbf{A}^{(i,j,k,m)}$ is determined by a functional form that is randomly distorted differently in each cell (see equations A4 and A5). These local distortions have been implemented in order to improve the statistical homogeneity of the turbulent field $B$.

The amplitude factors $a^{(i,j,k,m)}$ of magnetic fluctuations are determined considering the phenomenology of the turbulent cascade. Here, we assume that the turbulent field has a Kolmogorov spectrum, where the spectral energy density is $e(k) \propto k^{-5/3}$. This implies that the mean fluctuation at the scale $\ell$ is $\delta B(\ell) \propto \ell^{1/3}$. Although in a stationary situation the mean energy transfer rate ($\epsilon$) at a given spatial scale $\ell$ is independent of $\ell$, the local energy transfer rate $e$ is not spatially uniform, but can change from place to place according to the effectiveness of non-linear couplings (e.g. Frisch 1995). As a result, the amplitude of fluctuations is not spatially uniform, but fluctuations stronger than the average form, separated by regions where fluctuations are weaker. This feature propagates to smaller scales through a multiplicative process, becoming more and more relevant with decreasing $\ell$. Thus, at the smallest scale the field is characterized by very strong and localized fluctuations with wide ‘quiet’ regions in between: this is the phenomenology of intermittency.

In our STM, such a process is modelled as in the $p$-model by Meneveau & Sreenivasan (1987), where $p$ is a fixed parameter chosen in the interval $[1/2, 1]$. Energy flows from large to smaller eddies with an unequal rate $\epsilon$: each ‘parent’ eddy at a scale $\ell_m$ gives energy to its eight ‘daughter’ eddies at the scale $\ell_{m+1}$ with a rate $\epsilon_m + 1 = 2p\epsilon_m \geq \epsilon_m$ for four randomly chosen daughter eddies and $\epsilon_m + 1 = 2(1-p)\epsilon_m \leq \epsilon_m$ for the remaining four daughter eddies.
For $p = 1/2$, the rate $\epsilon$ is equal at all the scales and positions, $\epsilon_{m+1} = \epsilon_m$; this corresponds to a non-intermittent fluctuating field. For increasing $p$, differences between the rates increase, enhancing the level of intermittency. In our STM, $p$ is a tunable parameter that we use to investigate the effects of intermittency. In conclusion, the amplitude factors appearing in equation (3) are determined by

$$a^{(i,j,k,m)}_q = \sigma^{(i,j,k,m)}_q d_0 \frac{\ell_m^i \ell_m^j \ell_m^k \ell_m^l}{\epsilon_0^m}^{1/3}. \quad (4)$$

In equation (4), the parameter $d_0$ is used to regulate the standard deviation of the turbulent field; $\epsilon_0$ is the energy transfer rate at the scale $\ell_0$; the quantity $\sigma^{(i,j,k,m)}_q$ represents the randomly chosen sign of the eddy ($\sigma^{(i,j,k,m)}_q = \pm 1$).

The turbulent magnetic field $B$ is calculated through the following algorithm. In principle, the form (3) should be used, in which the sum can include a huge number of terms: for instance, to obtain a spectrum with a resolution of $10^4$ terms, one would need to sum at any position $x$ every eddy in the sum (5).

$$B(x) = B_0 + \sum_{m=m_{\text{min}}}^{m_{\text{max}}} \sum_{\mu=1}^{8} \delta B^{(i,x,m)}(x). \quad (5)$$

The above sum contains a much smaller number of terms than the sum in equation (3) (e.g. 112 instead of $\sim 10^{15}$ terms, for $N_s = 16$ and $m_{\text{min}} = 2$), which can be calculated with a small computational effort. To determine the magnetic field $B^{(i,x,m)}$ of the involved eddies, a set of parameters must be chosen. These are: (i) $\omega^{(i,x,m)}_0$, which defines the distortion of each eddy (equation A4); (ii) the sign $\sigma^{(i,x,m)}$ of each eddy (equation (4)); (iii) $\beta^{(i,x,m)}_0$ which defines the energy transfer rate of each eddy in terms of the rate of its parent eddy (equation A6). Such parameters are randomly determined through a random number generating routine, in which the initial seed is univocally chosen by the algorithm, as soon as the location $(i, j, k; m)$ within the lattice of the eddy ($\mu; x; m$) is known. Thus, the values of parameters (i)−(iii) defining eddies in the sum (5) are univocally determined as functions of the spatial position $x$. On the other hand, due to the very large number $N_{\text{cell}}$ of eddies contained in the model, the global randomness of those parameters is ensured.

Summarizing, the main features characterizing the STM are the following: (a) the evaluation of the field is very fast, due to the low number of terms to be summed at any position $x$; moreover, the computation time is proportional to the number of scales $N_s$; e.g. increasing $N_s$ by a factor of 2 would increase the spectral range by a factor of $2^{\epsilon_0}$ while the computation time would simply be increased by a factor of 2. (b) The model does not require large memory storage. Indeed, in the above-described algorithm, nothing needs to be kept in memory: each time the field needs to be calculated at a point $x$, this is done deducing all the properties of the involved eddies directly from the position $x$. (c) The turbulent field $B$ is continuous with all space derivatives up to the third order. (d) At variance with other methods, no spatial grids are involved: the field is directly calculated at any spatial point without any interpolation procedure.

In Fig. 1, the magnetic field power spectral density is plotted, showing the typical Kolmogorov-like scaling observed in the solar-wind inertial range. This spectrum has been generated by tracing a virtual satellite within the simulation box, so that the magnetic field has been ‘measured’ along a specific path. The spectrum in Fig. 1 refers to the case of $p = 0.7$; notice that in this model the magnetic field spectrum becomes steeper as the intermittency level, i.e. $p$, increases (not shown).

### 3 Numerical Simulations and Results

In order to describe the influence of turbulence on particle transport, we perform test particle simulations integrating the Lorentz equations for the trajectories and velocities of $N_p = 1000$ protons of fixed energy $\sim 1$ MeV using the Boris method as time stepper. Given the position $x_i$ and the velocity $v_i$ of a particle at the $i$-th step, the Boris algorithm computes the quantities at the $(i+1)$-th step by solving the following equations:

$$x_{i+1} = x_i + \frac{v_i}{\Delta t} \quad (6)$$

$$v_{i+1} = v_i + \frac{q}{m} \left[ E_i + \frac{(v_{i+1} + v_i) \times B_i}{2c} \right], \quad (7)$$

where $\Delta t$ is the step size, $x_i \equiv x(t_i)$, $v_i \equiv v(t_i - \Delta t/2)$, $t_{i+1} \equiv i\Delta t$, $E_i \equiv E(x_i, t_i)$, and $B_i \equiv B(x_i, t_i)$. The scheme can be made explicit solving analytically $v_{i+1}$ in terms of $v_i$. In the case under consideration here, $E_i = 0$. It has been shown that the method is symplectic and, in absence of an electric field, conserves the energy up to the round-off error (Webb 2014). Even though the conservation of energy is embedded in the method, a phase error may arise in resolving particle gyration around the magnetic field (Qin et al. 2013). In order to verify if this can influence diffusion, we tested the technique by comparing it with a fifth-order adaptive Runge–Kutta method (Press 2007). The evolution of the diffusion coefficients using the two methods does not show significant differences, while the Boris method provided better energy conservation and much shorter computational time. The Boris method can thus be safely used for particle motion integration. In every simulation, particles are placed with equal speed $v = 2 \times 10^4$ km s$^{-1}$ and random direction and position inside a cubic box of edge $L = 4L_0$. The magnetic
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Field is computed at each step through the STM described in the previous section and has this analytic form:

\[ \mathbf{B}(x) = B_0 \hat{z} + \delta \mathbf{B}(x), \]

where \( B_0 = 10^{-4} \text{G} \) and \( \delta \mathbf{B}(x) \) represents a 3D isotropic turbulent field with zero mean which is given by the fluctuating part of the magnetic field specified in equation (5). In this configuration, we can define the diffusion coefficients parallel and perpendicular to the mean magnetic field as

\[ D_{\parallel}(t) = \frac{1}{2 N_p l} \sum_{i=1}^{N_p} (z_i(t) - z_i(0))^2 \]

\[ D_{\perp}(t) = \frac{1}{2 N_p l} \sum_{i=1}^{N_p} \{[x_i(t) - x_i(0)]^2 + [y_i(t) - y_i(0)]^2\}, \]

where \( r_i(t) = (x_i(t), y_i(t), z_i(t)) \) is the position of the \( i \)-th particle in function of time. Using the simulations, we investigate how transport is affected by three different parameters of the STM: the spectral width \( \xi = l_c/l_{\text{min}} \); the amplitude of the turbulence \( \delta b = \delta B/B_0 \) (here \( \delta B \) is the magnetic field standard deviation computed on a 512\(^3\) grid of volume \( \Gamma \), controlled by properly setting the parameter \( a_0 \) in equation 4); the level of intermittency, identified by the parameter \( p \) of the \( p \)-model.

3.1 Varying the spectral width

In the simulations discussed in this section, we consider a large amplitude turbulence configuration (\( \delta b = 1 \)), typical of heliospheric plasmas. First, we set up \( p = 0.5 \), i.e. no intermittency. We have defined \( \xi = l_c/l_{\text{min}} \) as the spectral width and we can vary this parameter varying the smallest scale of the STM. To include the energetic proton Larmor radius \( R_L \) within the range of spatial scales of the STM spectrum, the condition \( \xi > 1/\rho \) must be satisfied. As mentioned in previous section, the above condition implies that for \( \xi > 250 \) the Larmor scale is inside the STM scale range, and it is outside that range for \( \xi < 250 \). We performed simulations with \( \xi \) either smaller or larger than 250. In Fig. 2, the diffusion coefficients in the direction parallel and perpendicular to the mean magnetic field as a function of time are plotted for \( \xi \) ranging from 4 to \( \xi \approx 16000 \). We can notice how the value of \( \xi \) actually affects transport. When the Larmor scale is much smaller than the smallest scale of the spectrum (\( \xi = 4 \)), the diffusion is enhanced both in parallel and in perpendicular direction. In this case (red curve), a regime of normal diffusion is reached for parallel transport at a time \( t \sim 10^3 \text{s} \), while subdiffusion is observed in the perpendicular direction. If the minimum scale approaches the Larmor scale, both diffusion coefficients become smaller.

We also obtain that the normal diffusion regime in the parallel direction is reached earlier, at a time \( t \sim 10^3 \text{s} \), and that in the perpendicular direction normal diffusion is recovered for longer times, while a subdiffusive regime can be identified for \( 10^3 \text{s} < t < 10^4 \text{s} \) (see the discussion below). Moreover, when the spectrum extends far beyond the Larmor scale, the diffusion coefficients barely change, meaning that all the fluctuations at a scale smaller than the Larmor radius are averaged over a gyration (orange and grey curves), see also Pommois et al. (2007). The asymptotic value of the diffusion coefficients has been obtained, for the cases of normal diffusion, by fitting the results of the mean square displacement for the last 500,000 s. The behaviour of \( D_{\parallel} \) and \( D_{\perp} \) as a function of \( \xi \) is shown in Fig. 3. It can be seen that for \( \xi > 128 \), the value of the diffusion coefficients does not change anymore.

Figure 2. \( D_{\parallel} \) and \( D_{\perp} \) as a function of time for different spectral widths (see legend).

Figure 3. Parallel (filled circles) and perpendicular (open circles) diffusion coefficients as a function of \( \xi \) averaged on the final \( 5 \times 10^5 \text{s} \).
of the trajectory of a single particle are shown for $\xi$. The PDF in this case has a maximum at $\tau \sim 10^2$ s and then decreases approximately following a power law with an exponent $\sim -2.36$. The green curve, corresponding to $\xi = 16$, has an exponent intermediate between $-1.4$ and $-2.36$. All the other cases at higher $\xi$ present similar PDFs of $\tau$. For these cases, it is possible to identify a power law with exponent $\sim -1.4$ that has a break at times $10^2 < \tau < 10^3$ s. We notice that the break location corresponds to the time when the normal diffusion regime sets up in parallel transport and it moves to smaller $\tau$ as $\xi$ increases. In Fig. 5, two samples of the trajectory of a single particle are shown for $\xi = 4$ and $\xi = 1024$. The plots show how trajectories and pitch-angle scattering change qualitatively when the condition $\xi > 1/\rho$ is satisfied. We selected two samples where multiple events of pitch-angle scattering are present. The left-hand panel of Fig. 5 shows that in the $\xi = 4$ case, the particle scatter is due to magnetic mirroring. On the contrary, in the case $\xi = 1024$ (right-hand panel), the pitch angle scattering events are concentrated in particular sites, possibly associated with magnetic mirroring as well. When a particle encounters one of such sites, it goes through multiple pitch-angle scattering or reflections; it eventually exits the site and ‘flies’ until it is not trapped again within another site. In the right-hand panel, the gyromotion is scarcely visible because the Larmor radius is small compared the scale of the plot, which shows the guiding centre motion. Both pitch angle scattering due to resonant interactions and magnetic mirroring can be recognized.

Figure 4. PDF of the inversion time $\tau$ for different spectral widths and corresponding power-law fits (see legend).

In Fig. 4, the Probability density functions (PDFs) of the inversion time $\tau$, i.e. the elapsed time between two sign inversions of the local pitch-angle cosine, are plotted for different $\xi$, being $\delta b = 1$. We notice that for $\xi = 4$ the inversion times extend up to $\tau \sim 10^3$ s, which is also the time necessary to reach normal diffusion in the parallel direction. The PDF in this case has a maximum at $\tau \sim 10^2$ s and then decreases approximately following a power law with an exponent $\sim -2.36$. The green curve, corresponding to $\xi = 16$, has an exponent intermediate between $-1.4$ and $-2.36$. All the other cases at higher $\xi$ present similar PDFs of $\tau$. For these cases, it is possible to identify a power law with exponent $\sim -1.4$ that has a break at times $10^2 < \tau < 10^3$ s. We notice that the break location corresponds to the time when the normal diffusion regime sets up in parallel transport and it moves to smaller $\tau$ as $\xi$ increases. In Fig. 5, two samples of the trajectory of a single particle are shown for $\xi = 4$ and $\xi = 1024$. The plots show how trajectories and pitch-angle scattering change qualitatively when the condition $\xi > 1/\rho$ is satisfied. We selected two samples where multiple events of pitch-angle scattering are present. The left-hand panel of Fig. 5 shows that in the $\xi = 4$ case, the particle scatter is due to magnetic mirroring. On the contrary, in the case $\xi = 1024$ (right-hand panel), the pitch angle scattering events are concentrated in particular sites, possibly associated with magnetic mirroring as well. When a particle encounters one of such sites, it goes through multiple pitch-angle scattering or reflections; it eventually exits the site and ‘flies’ until it is not trapped again within another site. In the right-hand panel, the gyromotion is scarcely visible because the Larmor radius is small compared the scale of the plot, which shows the guiding centre motion. Both pitch angle scattering due to resonant interactions and magnetic mirroring can be recognized.

beside the spectrum extension: in Fig. 6 we report the diffusion coefficient for $\xi = 4$ and $\delta b = 0.5$, a turbulence level which is also frequently observed in the solar wind. While the perpendicular diffusion coefficient reaches a constant value for times longer than $10^3$ s, we can see that the parallel diffusion coefficient does not saturate. This implies that the asymptotic regime has not yet been reached, due to the weakness of the interaction between particles and inhomogeneous fields. No resonant wave particle interaction is present in this case: particle motion is almost scatter-free, since the Larmor radius is smaller than the smallest scale of the magnetic field spectrum and the fluctuation level is lower than in the former runs. For parallel transport, a nearly ballistic regime is obtained up to about $10^3$ s. After that time, the growth of the mean square displacement is no longer quadratic in time, but still faster than linear. This means that from $t = 10^3$ s to $t = 10^4$ s superdiffusion is found, in agreement with previous studies by Pommois et al. (2007). Although one may argue that a truly time asymptotic regime has not been reached because a bending in the value of $D_t$ is visible, we should also consider that a time of $10^3$ s is very long in the heliosphere: therefore, for heliospheric energetic particles this result corresponds to an effective superdiffusive regime. We also notice that normal diffusion for perpendicular transport, obtained for $t > 10^3$ s, is mostly due to field line random walk, because $\nabla B$ drift is very small in this run and particles are streaming ‘scatter-free’ along the field lines at least up to $t = 10^4$ s.

3.2 Varying the amplitude of the turbulence

In the simulations described in this section, we keep the Larmor scale inside the spectrum setting $\xi = 1024$ and consider a non-intermittent turbulence ($\rho = 0.5$). The amplitude of the tur-
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Figure 6. $D_\parallel$ and $D_\perp$ for $\xi = 4$ and $\delta b = 0.5$ (red lines). For comparison, the results for $\delta b = 1$ are also shown (black lines).

Figure 7. $D_\parallel$ and $D_\perp$ as a function of different level of the turbulence (see legend).

Figure 8. Parallel (black filled circles) and perpendicular (black open circles) diffusion coefficients as a function of $\delta b$ averaged on the final $5 \times 10^5$ s. Parallel (red filled squares) and perpendicular (red open squares) diffusion coefficients in Giacalone & Jokipii (1999).

bulence is controlled by the parameter $\delta b$. In Fig. 7, $D_\parallel$ and $D_\perp$ as a function of time are shown for different $\delta b$. In every case, a diffusive regime is eventually reached in the parallel direction. We notice that this happens at later times when the level of the turbulence is the lower. The asymptotic value of the coefficient decreases when $\delta b$ increases, suggesting that strong turbulence reduces parallel transport. We can compare the scaling of the obtained diffusion coefficient with that reported by Giacalone & Jokipii (1999). Fig. 8 shows $D_\parallel$ and $D_\perp$ as a function of the turbulence level, together with the values of Giacalone & Jokipii (1999). The parallel diffusion coefficient $D_\parallel$ decreases for increasing $\delta b$, because of the enhanced pitch-angle scattering. On the contrary, the perpendicular diffusion coefficient $D_\perp$ increases with $\delta b$, because of the enhanced field line random walk. The two trends with $\delta b$ obtained for both coefficients are very similar to the results of Giacalone & Jokipii (1999). However, the diffusion coefficients in our simulations are in general smaller, up to one order of magnitude. Such discrepancy may be due the considerably larger proton energy assumed in the relevant runs by Giacalone & Jokipii (1999, 31.6 MeV), as compared to the cases presented here (2 MeV). Clearly, a larger energy implies a larger speed and hence a larger diffusion coefficient.

In the perpendicular direction, the situation is more complex. In the $\delta b = 1.0$ case, there is an indication of recovery of diffusion at
long times. On the contrary, for $\delta b = 0.2$ the behaviour is clearly subdiffusive at long times. An intermediate behaviour is obtained for $\delta b = 0.75$ and $\delta b = 0.5$, where the subdiffusive behaviour increases with decreasing $\delta b$. These results can be understood in terms of compound diffusion, a basically subdiffusive regime due to particles tracing back their field lines after pitch-angle diffusion (Webb et al. 2006; Pommois et al. 2007; Bitane, Zimbardo & Velti 2010; Tautz & Shalchi 2010; Giacalone 2013). In such case, the perpendicular spreading of particles is reduced, since particles perform an effectively anti-persistent random walk (Perrone et al. 2013; Zimbardo et al. 2015), which leads to subdiffusion. On the other hand, when the level of turbulence increases, the stochastic separation of magnetic field lines increases too, to the point that pitch-angle scattered particles do not trace back the ‘original’ field line, but some different field line which is diverging exponentially. This divergence is quantified by the Kolmogorov entropy and by the Kubo number (Rechester & Rosenbluth 1978; Zimbardo et al. 2008; Bitane et al. 2010), and can be shown to depend on the turbulence level and anisotropy. Therefore, for large turbulence levels and for magnetic fluctuations with enough perpendicular structure, normal diffusion is recovered (e.g. Qin et al. 2002a,b). In Fig. 9 the PDF of $\tau$ is plotted. As $\delta b$ decreases the PDFs extend to larger values of $\tau$. A power law of the type $\tau^{-1.4}$ fits the PDF for small values of $\tau$ for $\delta b = 1.0$. For smaller value of $\delta b$, the power-law range is too reduced to allow a conclusive fit.

3.3 Varying the level of the intermittency

The last part of the numerical investigation is related to the role of intermittency on transport. For this purpose, we fix $\xi = 16\,384$ in order to keep the Larmor scale always inside the STM inertial range, and we consider a large amplitude turbulence ($\delta b = 1$). In Section 2, we have explained how intermittency is reproduced through a $p$-model. We recall that in our model $p = 0.5$ means no intermittency and increasing value of $p$, represents stronger intermittency. We point out that $p = 0.7$ is considered a value appropriate to the ‘undisturbed’ solar wind, and that Yordanova et al. (2008) observed enhanced intermittency downstream of the Earth’s bow shock corresponding to $p$ values up to 0.96. In Fig. 10, the evolution of diffusion coefficients as a function of time is plotted for different values of $p$. In the parallel direction, a normal diffusion regime is reached at a time $t = 10^3$ s for all $p$ but, interestingly, the asymptotic value of the diffusion coefficients increases with the intermittency. In the perpendicular direction, changing the value of $p$ does not have an appreciable influence on the diffusion coefficient: the transport appears to be subdiffusive up to $10^5$ s and becomes eventually diffusive, with all the curves converging to nearly the same asymptotic value, although a slight decrease of $D_\perp$ with $p$ can be seen (see Fig. 11). A possible explanation for this enhanced parallel diffusion can be found in Fig. 12, where the PDFs of the inversion times $\tau$ are plotted. The shape of the PDFs does not change qualitatively as $p$ varies: they all present a power-law dependence of the type $\tau^{-1.4}$ at small $\tau$ followed by a sharp decay at larger $\tau$. However, the power-law break shifts to larger $\tau$ as $p$ increases. This suggests that intermittency increases the probability that a particle travels for longer time without reversing its direction of motion. For stronger intermittency, bursts of magnetic energy have larger amplitude and are non-homogeneously distributed in space, allowing particles to travel long distances between intermittent structures without being deviated from their direction of motion. Fig. 12 shows, for the first time, that larger $p$ result in more extended power-law distribution of the inversion times, before the break is reached.

It is worth mentioning that recently Perri & Zimbardo (2012b) have explored the pitch-angle scattering times (roughly corresponding to the inversion times) of energetic particles in the interplanetary
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Figure 11. Parallel (filled circles) and perpendicular (open circles) diffusion coefficients as a function of $p$ averaged on the final $5 \times 10^5$ s.

Figure 12. PDF of the inversion time $\tau$ for different levels of the intermittency (see legend).

Figure 13. PDF of the scattering times as computed in equation (10) from the magnetic field variances in the undisturbed solar wind as measured by Ulysses. A power-law fit is also plotted (full line).

at the Alfvén speed, that is almost one order of magnitude lower than the solar-wind bulk speed $V_{SW}$, we can consider the fluctuations frozen into the flow (Taylor hypothesis), so that time variations can easily be transformed into spatial variations, i.e. $\Delta r = V_{SW} \Delta t$ (e.g. Perri & Balogh 2010). The distributions of the scattering times found for $\sim 100$ keV particles are quite broad (see fig. 4 in Perri & Zimbardo 2012b), owing to the ‘burstiness’ of the magnetic field variances (strong intermittency). For comparison with the present simulations where the magnetic field parameters are close to those at 1 au, Fig. 13 shows the PDF of the scattering time in equation (10) in the fast wind computed using magnetic field data from the Ulysses spacecraft at 1.5 au (Balogh et al. 1992). The time average in equation (10) has been computed over about 7 min, corresponding to the scale of the Larmor radius of 1 MeV particles moving in a mean field of amplitude of 3 nT. The distribution, similarly to what has been found in Perri & Zimbardo (2012b), exhibits a power-law tail at large scattering times, implying that there is a non-negligible probability of large scattering times (due to regions of very low magnetic field variance). Both Figs 12 and 13 exhibit power-law distributions of either the inversion times or the scattering times: even if the exponents are rather different, these distribution show that the particle random walk in the direction parallel to $B_0$ is not Gaussian and is characterized by a broad distribution of scattering times and free-path lengths. This finding emphasizes the multiscale nature of particle transport in collisionless plasmas. However, the distribution plotted in Fig. 13 should not be directly compared with the ones reported in Fig. 12, since in the former case scattering times have been estimated from the values of the magnetic field at the spacecraft position, while in the latter case Lagrangian particle trajectories have been traced in the simulation box. Obviously, the Lagrangian approach cannot be adopted in spacecraft data. The comparison has been made to address the presence of a broad scattering time distribution in the solar wind due to the intermittency in the magnetic field. A deeper comparison between simulations and observations will be presented in a forthcoming paper.

4 DISCUSSION AND CONCLUSIONS

In this paper, we considered the energetic particle transport in the presence of magnetic turbulence. This problem has been studied
by means of a new realization of isotropic synthetic magnetic turbulence with large spectral extension and in which the level of intermittency can be varied. Typical values of the parameters have been set in accordance with the solar wind turbulence, that is, a correlation length for the magnetic turbulence corresponding to $5 \times 10^6$ km and a spectral index in the inertial range equal to $5/3$. Turbulence levels of the order of 0.5–1 have been used. An extensive set of numerical runs has been carried out by injecting monoenergetic particles, corresponding to $\sim 1$ MeV, in a simulation box where both the synthetic magnetic turbulence and a background magnetic field of 10 nT are present.

The main results can be summarized as follows.

(1) By varying the extension of the spectrum $\xi$, we found that both $D_\parallel$ and $D_\perp$ decrease with the increase of $\xi$ up to about 128; then, both $D_\parallel$ and $D_\perp$ attain a value independent of $\xi$ when the shortest fluctuation wavelength $\lambda_{\text{min}}$ is much smaller than the ion Larmor radius. The STM used for our simulations was crucial to achieve these results, because of its capability of reproducing a large inertial range extension turbulence with small computational costs.

(2) Inspection of the particle trajectories and of the particle inversions of parallel velocity shows that for a short spectrum with $\lambda_{\text{min}} > R_L$, the inversion of parallel velocity is mostly due to magnetic mirroring. For a longer spectrum, both pitch angle scattering and magnetic mirroring can be recognized, in agreement with the study of Goldstein (1976). For this result, it is essential to have the capability of varying the spectral extension up to very long spectra.

(3) By varying the turbulence level $\delta b$ from 0.2 to 1, we have found that $D_\parallel$ decreases with the increase of $\delta b$, while $D_\perp$ increases with the increase of $\delta b$. Also, the ratio $D_\parallel/D_\perp$ is of the order of 10–100. These results are in agreement with previous findings by Giacalone & Jokipii (1999), Casse et al. (2001), Hussein & Shalchi (2016), and others. On the other hand, we did not explore the cases with $\delta b \gg 1$, for which a smaller ratio $D_\parallel/D_\perp$ is expected.

(4) Anomalous, non-diffusive transport is found in some cases: in the case of short spectrum and $\delta b = 0.5$, superdiffusion in the parallel direction is obtained, in agreement with previous studies by Zimbardo et al. (2006) and Pommois et al. (2007) for the case of isotropic turbulence. In the perpendicular direction, we found subdiffusion at intermediate times and normal diffusion for long simulation times. The subdiffusive regime depends on the low stochasticity of the field lines: with our isotropic turbulence model, stochasticity only depends on the turbulence level. When the intensity of the turbulent fluctuations is comparable to the intensity of the background magnetic field, $\delta b = 1$, the transport in the perpendicular direction is normal; instead, a subdiffusive regime in the perpendicular transport is found either when the level of turbulence is low or in the case of short spectrum. This result is analogous to the one presented in Hussein & Shalchi (2016), where a model of dynamical turbulence has been implemented with a slab spectrum and a level of $\delta b = 1$.

(5) Starting from the computed particle trajectories, we have obtained the probability distribution functions of the parallel velocity inversion times, and have found that these distributions are power laws over more than one decade, and then decay. Such broad distributions imply that there is not a typical scattering time, at variance with expectations based on the quasi-linear approach. The possibility to have both short and long scattering times during parallel transport may help to understand the finding of short acceleration times for energetic particles at heliospheric shocks reported by Perri & Zimbardo (2015).

(6) Another novel feature of the STM model presented here was the possibility to reproduce intermittency through a $p$-model. We have shown here for the first time in a three-dimensional simulation the effects of intermittency on particle transport. In particular, we found that an increasing level of intermittency enhances parallel transport. Stronger intermittency implies the presence of more intense magnetic bursts, which are also highly localized in space. Particles can thus have longer parallel displacement between two successive interactions with such magnetic structures. This is reflected in the broadening of the distribution of the particle inversion times for increasing values of the parameter $p$. We did not observe any effect of intermittency on the perpendicular transport.

(7) We also computed the distribution of pitch angle scattering times, calculated in the quasi-linear theory framework using solar wind magnetic field time series detected by the Ulysses spacecraft at 1.5 au, for protons of 1 MeV. This distribution is also very broad and has a power-law tail, in a way similar to the distributions of inversion times found from the integration of particle trajectories. However, the power-law exponent is different from that of the distribution of inversion times. This suggests that further investigation of the statistical distribution of scattering times, obtained both from the numerical simulation and the solar wind in situ data, is needed.

Since for most of the simulations presented here a normal transport has been found along the parallel direction (except for $\delta b = 0.5$ and $\xi = 4$), it is interesting to give an estimate of the parallel mean free path, $\lambda_\parallel = 3D_\parallel/\nu$, and compare it with the values derived from observations, e.g. the Palmer consensus which corresponds to $\lambda_\parallel \approx 0.08–0.3$ au (Palmer 1982; Bieber et al. 1994). Recently, Hussein & Shalchi (2016) have shown that using a dynamical magnetic turbulence with a slab/2D spectrum and $B/B = 0.5$, a better agreement between mean free paths from simulations and observations is recovered. Indeed, numerical simulations and quasi-linear models have always underestimated $\lambda_\parallel$ (Bieber et al. 1994). Thus, from our simulations we get $D_\parallel \in [10^{20} - 10^{21}]$ cm s$^{-1}$ (see Figs. 1, 5, 7) and for particles of $\sim$44 MV rigidity (the ones injected in the system) the mean free path is $\lambda_\parallel \in [0.014 - 0.14]$ au. While 0.014 au is lower than the Palmer consensus, 0.14 au agrees reasonably well with it. It is interesting to notice that larger mean free paths are often deduced from the observations. For instance, studying helium pick-up ions with SoHO spacecraft at 1 au, Saul et al. (2007) estimated $\lambda_\parallel \approx 0.1–1.2$ au. Notice that $\lambda_\parallel$ can be higher in the case of short spectrum, namely $\xi = 4$ in our simulations (see Fig. 4).

A more detailed comparison between the model results and observations is deferred to a forthcoming study. Further, we plan to explore the influence of intermittency on the perpendicular transport when low level of turbulence is considered. The case of anisotropic turbulence has further to be taken into account. In fact, it has been shown that in astrophysical plasmas, for example in solar wind, the turbulence spectrum is highly anisotropic (Dasso et al. 2005; Horbury, Forman & Oughton 2008), so that the turbulent cascade favour the formation of smaller scales in the direction perpendicular to the local magnetic field. In conclusion, we believe that the approach used for this work can be a valuable tool for further investigation, and to shed light on the process of particle transport in astrophysical plasmas.

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APPENDIX A: DETAILS OF THE SYNTHETIC TURBULENCE MODEL

In this appendix, we quantitatively specify some features of the STM. Given the spatial domain \( \Gamma = \{ (x, y, z) \in [0, L_x] \times [0, L_z] \times [0, L_y] \} \), the explicit form of the cells is

\[
C^{(i,j,k,m)} = \{ (x, y, z) \in \left[ \left( i - \frac{1}{2} \right) \frac{L_x}{2^m}, \left( i + \frac{1}{2} \right) \frac{L_x}{2^m} \right] \times \left[ \left( j - \frac{1}{2} \right) \frac{L_y}{2^m}, \left( j + \frac{1}{2} \right) \frac{L_y}{2^m} \right] \times \left[ \left( k - \frac{1}{2} \right) \frac{L_z}{2^m}, \left( k + \frac{1}{2} \right) \frac{L_z}{2^m} \right] \},
\]

where \( m = 0, \ldots, N \) identifies the scale \( \ell_m \) and \( N \) is the number of scales. The indexes \( i, j, k = 1, \ldots, 2^m \) identify the cell position within the 3D lattice of cells at the \( m \)-th scale. Cells at each scale \( \ell_m \) are parallelepipeds with the same aspect ratio as \( \Gamma \), which do not intersect, except at borders.

The subdomains \( \gamma^{(i,j,k,m)} \) are also parallelepipeds defined as

\[
\gamma^{(i,j,k,m)} = \{ (x, y, z) \} = \left[ \left( i - \frac{3}{2} \right) \frac{L_x}{2^m}, \left( i + \frac{1}{2} \right) \frac{L_x}{2^m} \right] \times \left[ \left( j - \frac{3}{2} \right) \frac{L_y}{2^m}, \left( j + \frac{1}{2} \right) \frac{L_y}{2^m} \right] \times \left[ \left( k - \frac{3}{2} \right) \frac{L_z}{2^m}, \left( k + \frac{1}{2} \right) \frac{L_z}{2^m} \right].
\]

The centre of \( \gamma^{(i,j,k,m)} \) coincides with the centre of the corresponding cell \( C^{(i,j,k,m)} \), but \( \gamma^{(i,j,k,m)} \) is larger than \( C^{(i,j,k,m)} \) by a factor of 2 along each space direction. Thus, adjacent subdomains partially overlap.

The explicit form of the normalized vector potential in each subdomain is

\[
A^{(i,j,k,m)}(x, y, z) = \frac{\ell_m}{\ell_0} F(\chi) F(\eta) F(\zeta),
\]

where \( F(t) \) is the following polynomial function which determines the spatial profile of the eddy

\[
F(t) = 256t^8 - 256t^6 + 96t^4 - 16t^2 + 1, \quad \text{for} \quad -\frac{1}{2} \leq t \leq \frac{1}{2}
\]

\[
F(t) = 0, \quad \text{elsewhere}
\]
The variables \( \chi \), \( \eta \) and \( \zeta \) are defined by

\[
\chi = \chi^{(2m)} + \alpha^{(i,j,k,m)}_\chi \left( \chi^{(2m)} \right)^2 - \frac{1}{4}
\]

\[
\eta = \eta^{(2m)} + \alpha^{(i,j,k,m)}_\eta \left( \eta^{(2m)} \right)^2 - \frac{1}{4}
\]

\[
\zeta = \zeta^{(2m)} + \alpha^{(i,j,k,m)}_\zeta \left( \zeta^{(2m)} \right)^2 - \frac{1}{4},
\]

(A4)

where \( \alpha^{(i,j,k,m)}_\chi \), \( \alpha^{(i,j,k,m)}_\eta \), and \( \alpha^{(i,j,k,m)}_\zeta \) are constants which are randomly chosen in the interval \([-1, 1]\), while \( \chi^{(2m)} \), \( \eta^{(2m)} \), and \( \zeta^{(2m)} \), each belonging to interval \([-1/2, 1/2]\), are rescaled local spatial coordinates defined within the subdomain \( \gamma^{(i,j,k,m)} \) by the relations:

\[
\chi^{(2m)} = X^{(2m)}(x) = \frac{2^{m-1}}{L_x} \left[ x - \left( i - \frac{1}{2} \right) \frac{L_x}{2^m} \right]
\]

\[
\eta^{(2m)} = Y^{(2m)}(y) = \frac{2^{m-1}}{L_y} \left[ y - \left( j - \frac{1}{2} \right) \frac{L_y}{2^m} \right]
\]

\[
\zeta^{(2m)} = Z^{(2m)}(z) = \frac{2^{m-1}}{L_z} \left[ z - \left( k - \frac{1}{2} \right) \frac{L_z}{2^m} \right].
\]

(A5)

The function \( F(t) \) has one single maximum at \( t = 0 \) \( [F(0) = 1] \) and vanishes with its derivatives up to the fourth order at \( t = \pm 1/2 \). Then, it represents a localized eddy which matches with neighbouring eddies with continuous derivatives up to the fourth order. The non-linear mapping (A4) introduces a distortion in the spatial profile of the eddy along the three spatial directions, whose entity is determined by the three random numbers \( \alpha^{(i,j,k,m)}_\chi \), \( \alpha^{(i,j,k,m)}_\eta \), and \( \alpha^{(i,j,k,m)}_\zeta \). The above regularity properties of the vector potential are preserved by the mapping (A4).

The relation between the spectral energy transfer rate \( \epsilon_m \) of a given parent eddy at the scale \( \ell_m \) and the rate \( \epsilon_{m+1,n} \) of its eight daughter eddies is given by the expression

\[
\epsilon_{m+1,n} = 2p \epsilon_m \beta_n^{(i,j,k,m)} + 2(1-p) \epsilon_m \left( 1 - \beta_n^{(i,j,k,m)} \right),
\]

\[
m = 0, \ldots, N_s, \quad n = 1, \ldots, 8 \quad \text{(A6)}
\]

where \( \beta_n^{(i,j,k,m)} = 1 \) for four randomly chosen daughters (for instance, \( n = 3, 5, 7, 8 \)) which receive more energy, while \( \beta_n^{(i,j,k,m)} = 0 \) for the remaining four daughters (for the same instance, \( n = 1, 2, 4, 6 \)) which receive less energy. For \( p = 1/2 \) we have \( \epsilon_{m+1,n} = \epsilon_m \), corresponding to a non-intermittent fluctuating field. With increasing \( p \) above the value 1/2, differences between the rates increase and the level of intermittency increases, as well.

The parameters: (i) \( \alpha^{(i,j,k,m)}_\chi \), defining the distortion of each eddy (equation A4); (ii) the sign \( \sigma^{(i,j,k,m)}_\chi \) of each eddy (equation 4); and (iii) \( \beta^{(i,j,k,m)}_n \), defining the energy transfer rate of each eddy in terms of the rate of its parent eddy (equation A6) are determined in the following way: for any given cell its integer absolute address \( l^{(i,j,k,m)} \) is calculated using the expression:

\[
l^{(i,j,k,m)} = i + (j - 1)2^m + (k - 1)2^m + f(m),
\]

(A7)

where \( f(m) \) is defined as follows:

\[
f(m) = \begin{cases} 
0 & \text{if } m = 0; \\
\sum_{n=0}^{m-1} 2^{2n} & \text{if } m \geq 1.
\end{cases}
\]

(A8)

The expression (A7) generates all the integers between 1 and \( N_{cell} \), thus defining a one-to-one correspondence between the set \( \{1 \leq l \leq N_{cell}, f \text{ integer}\} \) and the set of cells. The integer \( l^{(i,j,k,m)} \) is used as a seed for a random-number-generating routine, which produces the above parameters (i), (ii), and (iii). Since the total number \( N_{cell} \) of values of \( l^{(i,j,k,m)} \) is, in practice, extremely high (equation 1), a global randomness of the parameters defining the structure of single eddies is ensured.

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