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Linear and nonlinear regimes of bump-on-tail instability through Vlasov and toy model simulations

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Abstract – Resonant wave-particle interaction in collisionless unmagnetized plasmas is numerically investigated by means of a Fermi-like model, focusing on the linear and nonlinear regimes of the well-known bump-on-tail instability. Within this toy model, particle trapping effects are described through elastic collisions of particles with two barriers separated by a fixed length and whose amplitude (proportional to the wave energy) can increase or decrease in time, due to the sequence of stochastic collisions. The systematic comparison of the toy model numerical results with those obtained from Vlasov-Poisson simulations as well as with the predictions of kinetic theory, shows that the nonlinear map, on which the Fermi-like model is based, captures the basic physics of the linear growth of the bump-on-tail instability and of the particle trapping effects which produce the saturation of the instability and drive the nonlinear phase of wave-particle interaction.

The resonant wave-particle interaction in collisionless plasmas [1] is the basic process in many physical phenomena in space as well as in laboratory environments. In the framework of kinetic theory, resonant wave-particle interaction is described through Vlasov equation. This equation, which neglects binary collisions, retains collective particle interaction effects and provides physical mechanisms through which the particle distribution function can relax towards equilibrium. This relaxation process can arise, for example, through the growing of a small perturbation imposed on a stationary state.

When the distribution of particle velocities departs from the Maxwellian thermodynamic equilibrium and exhibits distinctly non-thermal features, it can be the source of free energy for many instabilities. The instability we will discuss in the present work is known as bump-on-tail instability [2] and arises from a gentle bump in the tail of a Maxwellian velocity distribution. The physical process that drives the linear growing of the perturbation amplitude is based on the resonant wave-particle interaction: a particle with velocity \( v \) close to the wave phase velocity \( v_p \) gains energy in each “collision” with the wave if \( v \lesssim v_p \), while it loses it if \( v \gtrsim v_p \); when the equilibrium distribution of particle velocities is such that \( (\partial f_0/\partial v)_{v_p} > 0 \), the wave amplitude grows exponentially in time.

More than forty years ago, O’Neil showed [3] that nonlinear effects, like particle trapping, are no longer negligible in wave-particle interaction processes, for times comparable to the so-called trapping time, i.e. the characteristic oscillation period of a charged particle in the potential well of a sinusoidal wave. For times comparable to the O’Neil trapping time, then resonant particles start oscillating in the wave trough, this causing a periodic energy exchange at each wave-particle “collision”. It can be shown [2] that this nonlinear trapping interaction causes the saturation of the bump-on-tail instability, by flattening the particle velocity distribution at \( v \simeq v_p \).

The O’Neil regime of wave-particle interaction has been deeply investigated more recently in both magnetized and
unmagnetized plasmas, in particular in the case of nonlinear Landau damping of electrostatic perturbations [4–9].

From the considerations above, one can argue that both linear and nonlinear regimes of the bump-on-tail instability are driven by the resonant exchange of energy between wave and resonant particles, while the bulk of the particle distribution does not play any significant role. The goal of the work presented in this paper is to show how a simple Fermi-like model, where wave-particle interaction is described as elastic collisions of particles with two barriers of variable amplitude, retains the fundamental physical aspects of the resonant energy exchange [9–11].

Monte Carlo simulations of this Fermi-like model have been carried out. In these simulations, velocities are normalized to the thermal velocity of plasma. Each particle is identified by the random initial position $x_j(0)$, uniformly distributed in the range $[-L/2, L/2]$, and by a constant speed $v_j$, extracted from the normalized distribution function $f_0^F(v,t=0) = \frac{1}{\sqrt{2\pi}} e^{-v^2_p/2(1+v v_p)}$ (the subscript $F$ indicates the particle distribution for the Fermi-like model simulations), where $-\sqrt{A_0} \leq v \leq \sqrt{A_0}$, with $A_0 = A(0)$. The numerical domain is given by $D_1 = [-L/2, L/2] \times [-\sqrt{A_0}, \sqrt{A_0}]$ with $A_0 = 1$. We use $N = 10^5$ particles, enough for an accurate statistics. Particles are placed between the barriers at different heights $y_j \leq A_0$, taken equal to their square velocity $y_j = v_j^2$. The dynamics of the $j$-th particle between two collisions is trivially described by the equation of motion of a point mass moving at constant speed $v_j$. For our simulations we choose $v_p = 1$. 

![Fig. 1: Panel (a1): time evolution of the electric-field amplitude in linear regime, from the Vlasov simulation; panel (a2) growth rate $\gamma_V$ vs. $Df_0^V$ (amplified by a factor $10^2$), from the Vlasov simulation; panel (b1): time evolution of the barriers amplitude in linear regime, from the Fermi-like simulation; panel (b2) growth rate $\gamma_F$ vs. $v_p Df_0^F$, from the Fermi-like simulation.](image-url)
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The numerical results of the above Fermi-like model will be compared with those obtained through a kinetic code [12,13] that numerically solves the Vlasov equation for the electron distribution function (ions are considered as a motionless background with constant density $n_0 = 1$) coupled to the Poisson equation for the self-consistent electrostatic field $E$, in a one-dimensional $(x,v)$ phase space (we assumed periodic boundary conditions in the spatial numerical domain). The numerical algorithm used to integrate the Vlasov equation is based on the well-known splitting method [14] for the time advance of the electron distribution function. In these kinetic simulations, times are normalized to the inverse electron plasma frequency $\omega_{pe}$ and lengths to the Debye length $\lambda_D$; as a consequence, velocities will be scaled by the electron thermal speed $v_{te}$. The initial distribution function for electrons is Maxwellian in velocities with a gentle bump in its tail, over which we superposed a spatial perturbation of small amplitude $\epsilon$ and wave number $k$ (we excited the maximum wavelength that fits in the numerical domain):

$$f_V^0 = f_V(x,v,t=0) = \{n_1 e^{-v^2/2} + n_2 [e^{-(v-V_0)^2/2} + e^{-(v-V_0)^2/2}] \times [1 + \epsilon \cos (kx)] \}$$(1)

(the subscript $V$ indicates the particle distribution for the Vlasov simulations) where $n_1 = 0.85$, $n_2 = 0.075$, $\epsilon = 10^{-5}$, $k = \pi/10$ and $V_0$ indicates the location of the bump in the tail of the velocity distribution. In our simulations, we typically have $V_0 = 4$. The limits of the velocity simulation domain are fixed at $v_{\text{max}} = 8$. Typically, each simulation is performed with $n_x = 512$ gridpoints in the spatial domain and $n_v = 1600$ gridpoints in the velocity domain. We choose a time step $\Delta t = 10^{-3}$, such that the Courant-Friedrichs-Levy stability condition is satisfied [15].

In fig. 1, we report the numerical results of both Vlasov simulation (panels $(a_1)$ and $(a_2)$) and Fermi-like model (panels $(b_1)$ and $(b_2)$), concerning the linear stage of the bump-on-tail instability. The panels $(a_1)$ and $(b_1)$ show, in semi-logarithmic plot, the time evolution of the electric-field amplitude (the spectral component $E_k$) from the Vlasov simulation and of the barriers amplitude from the Fermi-like model simulation, respectively. The typical linear growth is clearly visible in both plots. It is worth noting that the first trapping oscillation, signature of the nonlinear saturation of the instability, is also visible in panels $(a_1)$ and $(b_1)$. In order to make a more quantitative analysis of the linear stage of the bump-on-tail instability, we carefully compared the numerical results with the predictions from linear kinetic theory.

From Vlasov theory [2], the growth rate of the perturbation amplitude in the bump-on-tail instability is proportional to the velocity derivative of the equilibrium distribution of the particle velocity in the vicinity of the phase speed of the wave, $\gamma_V \propto \partial f_V^{eq} / \partial v \big|_{v_p}$ (from now on we will use the definition $D \equiv \partial / \partial v_p$). We then performed several Vlasov simulations, with different values of $V_0$, keeping fixed the other parameters and in particular the value of the wave number. As we discussed above, $V_0$ determines the position of the bump in the tail of the particle velocity distribution; therefore, varying $V_0$ produces a change in $Df_V^0$. The linear dependence of $\gamma_V$ on $Df_V^0$ is clearly visible in panel $(a_2)$ in fig. 1. As discussed in ref. [9], the expression for the growth rate of the barriers amplitude within the Fermi-like model can be easily found by conjecturing that the time variation of the wave energy must be proportional to the energy exchanged during each wave-particle collision divided by a typical transit time between barriers (see ref. [9] for details). From these simple considerations, one gets that the growth rate for the bump-on-tail instability within the Fermi-like model is $\gamma_F \propto \nu_p Df_F^{eq} = v_p^2 \exp(-v_p^2/2)$. The linear dependence of $\gamma_F$ on $v_p Df_F^{eq}$ is clearly visible in panel $(b_2)$ in fig. 1.

In the following, we discuss numerical results about the nonlinear saturation of the bump-on-tail instability. The top plot in fig. 2 shows the time evolution of the logarithm of the electric-field amplitude (the spectral component $E_k$) from the Vlasov simulation, while the bottom plot in the same figure shows the time evolution of the logarithm of the barriers amplitude from the Fermi-like model simulation. Both plots refer to the nonlinear stage of the system evolution. As we discussed above, when nonlinear effects start driving the system dynamics, the wave growing is stopped and the wave amplitude starts...
oscillating in time around a certain saturation level. In both plots of fig. 2, the linear stage of the bump-on-tail instability is followed by the nonlinear saturation, characterized by the occurrence of trapping oscillations in the wave amplitude.

O’Neil theory [3] predicts that the trapping time in nonlinear wave-particle interaction is inversely proportional to the square root of the wave amplitude ($\tau \propto A^{-1/2}$). We then repeated both Vlasov and Fermi simulations with different parameters in order to change the saturation level $A_{\text{sat}}$ of the wave amplitude and analyzed the dependence of the trapping time on $A_{\text{sat}}$. The trapping time is calculated as the characteristic time of the wave amplitude oscillations displayed in fig. 2. The results of this analysis are displayed in fig. 3. In the top panel of this figure we report $\tau_V$ (i.e. the trapping time from the Vlasov simulation) vs. the saturation amplitude of the electric field, while at the bottom we plot $\tau_F$ (i.e. the trapping time from the Fermi-like model simulation) vs. the saturation level of the barriers height. The numerical dots in both graphs are fitted with the curve $y = C(A_{\text{sat}})^{\beta}$ (where $C$ is a constant), obtaining the value $\beta \approx -0.493 \pm 0.011$ for the Vlasov simulation and $\beta \approx -0.509 \pm 0.023$ for the Fermi-like model simulation, in agreement with O’Neil predictions.

Nonlinear effects in resonant wave-particle interaction cause a distortion of the distribution function in the region around the wave phase speed and consequently alter the growth rate, since it is proportional to $(\partial f / \partial v)_p$. Therefore, the wave amplitude growing predicted by linear theory will be eventually stopped by nonlinear effects that alter the particle distribution function. As has been shown in the framework of *quasi-linear* kinetic theory [2], the wave amplitude growing in the bump-on-tail instability is stopped by a nonlinear flattening of the particle velocity distribution at $v \approx v_p$, due to trapping.

In the top plot of fig. 4, we show the electron distribution function (averaged on $x$) in the vicinity of the wave phase speed at $t = 0$ (black line) and at $t = 500$ (red line), from the Vlasov simulation; bottom: time history of the velocity derivative of the distribution function at $v \approx v_p$, from the Vlasov simulation.

We then repeated, for the Fermi-like model simulation, the same analysis described in fig. 4 for the Vlasov
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Fig. 5: (Colour on-line) Top: distribution of particle velocities in the vicinity of the wave phase speed at $t = 0$ (black line) and at $t = 40$ (red line), from the Fermi-like simulation; bottom: time history of the velocity derivative of the distribution function at $v \approx v_p$, from the Fermi-like simulation.

simulation. In the top plot of fig. 5, the black line indicates the distribution of particle velocities at $t = 0$, while the red line clearly shows the flattened velocity distribution at the end of the simulation. In the bottom panel of the same figure, we show the time evolution of the derivative of the velocity distribution at $v \approx v_p$ ($Df_F$). As can be easily seen, the velocity derivative of $f_F$ decreases in time due to nonlinear trapping and gradually approaches zero.

In this paper, we analyzed both linear and nonlinear regimes of the well-known bump-on-tail instability, by comparing the numerical results of a Fermi-like simulation, where particle trapping is modeled as a sequence of elastic collisions of a large number of particles with two barriers of variable height, with those obtained through Eulerian Vlasov-Poisson simulations. Moreover, the results of the numerical simulations of both Fermi-like model and Vlasov-Poisson code have been compared with the predictions of linear and quasi-linear kinetic theory. Our analysis shows that the Fermi-like model, based on a very simple particle dynamics, successfully captures the basic physical processes that drive the resonant wave-particle energy exchange and reproduces the phenomenology predicted within the kinetic theory and recovered in the Vlasov-Poisson simulations.

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