The approach to statistical equilibrium in collisionless wave-particle interactions

Vincenzo Carbone *, Rossana De Marco, Francesco Valentini, Pierluigi Veltri

Dipartimento di Fisica and Istituto Nazionale di Fisica della Materia, Università della Calabria,
Ponte P. Bucci Cubo 31C, 87036 Rende (CS), Italy

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Abstract

The long-time behaviour of the wave-particle interaction is investigated through a Fermi-like model that describes the interaction as the results of a large number of collisions between an ensemble of particles and two infinitely massive barriers with variable amplitude. The results are compared with those of particle in cell (PIC) simulations. The system approaches statistical equilibrium through a phase space filamentation that is analogous to a laminar Gibbs process in a conservative system.

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The wave-particle interaction process in plasmas is a basic phenomenon in space and laboratory settings because it is responsible for wave dissipation even in absence of collisions, according to the well known result by Landau [1]. For any monotonically decreasing distribution function, and in particular for a Maxwellian one, the wave is damped exponentially in time. Landau’s treatment is rigorous, but holds only for infinitesimally small initial perturbations. In nonlinear regime, the damping should be prevented [2] by trapping of particles in the potential well of the wave. In fact, for a finite initial amplitude of the electric field, trapped electrons both can gain and lose energy in the interaction with the wave, thus turning off Landau damping. The problem has been investigated numerically [3–7] by solving the Vlasov–Poisson set of equations, and only recently it has been understood that the framework of dynamical system can be useful to investigate the behaviour of particles in the wave field [6,8,10]. Numerical and analytical results [5,9] show the existence of a critical initial amplitude of the wave that marks the transition between the Landau scenario (for low initial amplitudes) and a scenario where the damping is prevented and the wave amplitude oscillates around an approximately constant saturation value. Some authors [7] questioned the physical meaning of these oscillations, on
the basis of the fact that a truly dissipationless simulation of the Vlasov–Poisson system cannot be made. As a consequence it is not clear how a collisionless plasma approaches the equilibrium.

In the present work we study wave-particle interaction, and in particular the approach towards equilibrium, by means of a Fermi-like model [10]. The results will be compared with those of a PIC simulation [11].

In the Fermi-like model [10] the wave particle interaction is represented as the result of a large number of collisions between an ensemble of \( N \) particles moving between two infinitely massive, one-dimensional barriers with variable length, set at a fixed distance \( L = 2 \). The whole system moves with speed \( v_{th} \) (which represents the wave phase velocity). In the barriers frame of reference, each elastic collision of a particle with a barrier will reverse the direction of motion of the particle, i.e. \( v' = -v \) (\( v' \) is the velocity after the collision). The length of the barriers \( A(t) \) represents the wave amplitude and the wave energy is defined as \( E_W = \frac{1}{2} A(t)^2 \). When a collision occurs the amplitude of the barriers varies according to \( A(t') = [A(t)^2 + \sigma(t)]^{1/2} \), where \( \sigma(t) = vv_{th}/v_{th} \) is the sign of the particle velocity before the collision and \( \Delta E = 4vv_{th} \) is the energy exchange between the particle and the barrier itself. Monte Carlo simulations of the model have been carried out. Velocities are normalized to the thermal velocity of plasma, i.e. \( u = u/v_{th} \) and \( u_p = v_{th}/v_{th} \). Each particle is identified by the random initial position \( x_j^{(0)} \) \( (j = 1, 2, \ldots, N) \), uniformly distributed in the range \([-L/2, L/2]\), and by a constant speed \( u_j \) extracted from the normalized distribution function \( f_0(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \), where \(-\sqrt{A_0} \leq u \leq \sqrt{A_0} \), with \( A_0 = A(0) \). The numerical domain is given by \( D_1 = [-L/2, L/2] \times [-\sqrt{A_0}, \sqrt{A_0}] \) with \( A_0 = 1 \). We use \( N = 10^5 \) particles, enough for an accurate statistics. Particles are placed between the barriers at different heights \( y_j \leq A_0 \), taken equal to their square velocity \( y_j = u_j^2 \). The dynamics of the \( j \)th particle between two collisions is trivially described by the equation of motion of a point mass moving at constant speed \( u_j \). For our simulation we choose \( u_p = 1 \). The results obtained with the model reproduces all the features of both Landau and O’Neil scenarios (for details, see Ref. [10]).

Particle-in-cell simulations [13] follow the electron dynamics in the \( x \)-direction, i.e. the direction of wave propagation (the ions are considered as a motionless background). The equations of motion of \( N = 10^7 \) electrons are integrated numerically through a standard leapfrog scheme. Time is scaled by the inverse plasma frequency \( \omega_p^{-1} \), where \( \omega_p = \sqrt{4\pi n e^2/m} \) is the electron density. Length is scaled by the Debye length \( \lambda_D = v_{th}/\omega_p \). With these choices, velocity is scaled by the electron thermal velocity \( v_{th} = \lambda_D/\omega_p \) and the electric field by \( \sqrt{4\pi n m v_{th}^2} \). The electron phase space domain for the simulations is \( D = [0, L_x] \times [-v_{max}, v_{max}] \) where \( v_{max} = 5 \). Periodic boundary conditions in physical space are imposed. The initial velocity distribution is a Maxwellian, over which a modulation in the physical space with amplitude \( A \) and wave vector \( k = 2\pi n/L_x \) is superposed: \( f(x,v,t = 0) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} [1 + A \cos(kx)] \). We perturbed the mode with the largest wavelength in the simulation domain. The length of the spatial domain is \( L_x = 20 \), the amplitude of the perturbation is \( A \approx 0.064 \). The simulations follow the evolution of electrons for many plasma periods \( (t_{max} = 4000) \).

The time evolution of the electric field spectral component \( E_y(t) \) from the PIC simulation is shown in the top plot of Fig. 1. At the bottom in the same figure, we report the amplitude of the barriers from the Fermi-like model as a function of time. In both plots, after a linear decay regime, the wave amplitude displays almost periodic oscillations in time, that become lower and irregular as time goes on. Even though this kind of oscillations have been shown in previous works [4–7], the long time regime, that recalls the O’Neil scenario for nonlinear Landau damping [2], has never been reached in numerical simulations. The oscillations of the wave amplitude described in Fig. 1 are driven by trapping of particles in the wave potential well, that is sinusoidal for the PIC simulation, and squared for the Fermi-like model. The particle dynamics is similar in both cases: particles whose energy is larger than the potential barrier free stream unperturbed; a few particles at the border of the trapped region can escape, generating flights of different extension before being retrapped. The bulk of particles, with energy below the potential barriers, are ever trapped.

Even if the behaviour of the trapped particles is trivial, they play an extremely important role. These particles follow a closed path in the phase space \((x,u)\). Each path (elliptical for the PIC simulation and rectangular for the Fermi-like model) represents a torus of the system, and particles move around this torus at different times, according to their speed. In the left panels of Fig. 2 is reported the phase space \((x,u)\) for the Fermi-like model. Each point in the scatter plots represents the position \( x \) and the velocity \( u \) of the \( j \)th particle at time \( t \). An analogous picture (not shown here) is obtained with the PIC code, (see [13]). As it can be seen from the figure, particle dynamics generates filaments in phase space, that become thinner and
thinner, up to a time where a complete mixing is reached. The particle distribution function can be defined through the usual coarse-graining of a point system in statistical mechanics, say \( f(x, u, t) \Delta x \Delta u \) represents the number of points that lie within the box of (narrow by finite) area \( \Delta = \Delta x \Delta u \). In right panels of Fig. 2 the electron distribution \( f(u, t) \) (averaged on \( x \)) is shown at different times form the Fermi-like model. From these plots we argue that the process of phase space filamentation corresponds to presence of velocity “stripes”. As time goes on, these stripes get narrower and disappear when the distribution becomes flat (when the total mixing stage is reached).

A quick reflection is enough to realize that the occurrence of velocity stripes produces a synchronization of collisions on each barrier. As the stripes get narrower, less and less particles collide almost synchronously on a given side of the potential well. Actually this is the dynamical process that is responsible for the presence of the amplitude oscillations shown in Fig. 1. In fact when the stripes are large enough, the synchronization of collisions on a given barrier is able to sustain or decrease the amplitude of the oscillations around the saturation value. When the filamentation goes on producing narrower velocity stripes, the synchronization of collisions is felt by a lower number of particles and the oscillations become less regular up to a complete phase-mixing. This means that, for long times, the system has reached the equilibrium configuration, and the approach to equilibrium is described by a filamentation of the phase-space. The approach of systems to statistical equilibrium through a laminar filamentation is analogous to the well known phenomenon of laminar mixing between two fluids in statistical mechanics that has been investigated by Gibbs. The Fermi-like dynamical system plays the role of an area-preserving two-dimensional map, as the well known bakers’ transformation [12].

This Gibbs filamentation that drives the system towards equilibrium ends up to a complete mixing when the width of the velocity stripes gets smaller than the characteristic scale of discreteness of the system (inter-particle separation). In Vlasov theory no characteristic length is present; as a consequence the filamentation process in Vlasov systems could keep going on and on producing thinner and thinner filaments in phase space, never reaching the complete mixing stage. This means that Vlasov theory cannot describe a Gibbs process.
With simple arguments we can better explain the approach towards equilibrium in terms of statistical mechanics concepts. In absence of Coulomb collisions, the trapped region can be thought as an isolated system with respect to the free region, where particles do not exchange energy with the wave (except for a few particles that can be trapped and detrapped by the wave more or less periodically in time). The statistical entropy of this system is defined as:

\[ S = \frac{1}{C_0} R D_s \log f \] 

where \( D_s \) is the width of the trapped region. It is straightforward to show that, imposing the entropy \( S \) to be maximum, the distribution function \( f \) turns out to be flat over the trapped region. As shown in Fig. 2, the final state of our Fermi-model simulation displays a flat velocity distribution. This means that the system naturally evolves towards the maximum entropy state.

We can describe the time evolution of the entropy for the Fermi-like model by evaluating the numerical entropy \( S_{\text{num}} \) on a discretized phase space. Fig. 3 shows time evolution of the entropy variation \( \Delta S_{\text{num}}(t) = S_{\text{num}}(t) - S_{\text{num}}(0) \) for three different values of \( N_g \) (the number of numerical cells in phase space). After the initial growing, \( \Delta S_{\text{num}}(t) \) reaches a saturation value, that does not depend on the numerical resolution. The jump in the entropy of the system corresponds to the entropy difference \( \Delta S_{\text{th}} \) between a flat velocity distribution function and a linearly decreasing one that fits our initial condition. The quantity \( \Delta S_{\text{th}} \) is indicated in the figure by a horizontal solid line.

As a last consideration, we numerically evaluate the two-particle correlation function. In collisionless systems, the dynamics of particles trapped in the wave field is usually described through the Vlasov equation. This is obtained from Boltzmann equation if one assumes that the two-particle correlations are negligible. In the interaction between particles and waves, long range correlations between particles are created by the wave
potential since the latter traps the particles and connects their trajectories in phase space. The normalized two-particle correlation function is defined as:

$$C_2(i,j) = \frac{f_2(i,j) - f_1(i)f_1(j)}{f_1(i)f_1(j)}$$

where $f_1$ and $f_2$ represent the single-particle and the two-particle coarse grained distributions respectively, the integers $i$ and $j$ indicate the $i$th and $j$th cells of area $\Delta$ in phase space. In Fig. 4 we show the time evolution of the maximum value of the two-particle correlation function $C_2$ from the PIC code and from the Fermi-like model. In both cases, the evolution of max($C_2$) is followed up to the time when the first two oscillations take place in the signal described in Fig. 1. The exponential damping regime in Fig. 1 corresponds to an increasing of the two-particle correlation function, that finally saturates when regular oscillations occur in Fig. 1. This shows that wave-particle interaction creates correlations in phase space particle trajectories.

Fig. 3. Time evolution of the entropy variation, for the Fermi-like model, for three different values of cells $N_g$ in phase space.

Fig. 4. Time evolution of the maximum of the two-particle correlation function $C_2$ in the resonant region, from the PIC code (top) and the Fermi-like model (bottom).
In conclusion, both PIC and Fermi-like simulations show that the system reaches the equilibrium through a laminar Gibbs process during which entropy and two-particle correlations increase in time.

References