Proper Orthogonal Decomposition of
two-dimensional turbulence in a pure electron plasma

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Abstract. The free-decaying two-dimensional (2D) turbulence in a pure electron plasma confined in the Malmberg-Penning trap ELTRAP is investigated experimentally and analyzed through the Proper Orthogonal Decomposition (POD). POD is used to extract coherent structures of the flow from a sequence of plasma density measurements, which represent the vorticity of the 2D fluid. The coherent structures that are energetically dominant are identified and their spatio-temporal dynamics is studied over the time evolution of turbulence. The results suggest the the dominant POD modes can be identified with diocotron modes which appear to be active during both the onset and relaxation phases of turbulence.

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INTRODUCTION

Highly magnetized pure electron plasmas confined in Malmberg-Penning traps provide the opportunity to experimentally investigate two-dimensional (2D) fluid dynamics in conditions where non-ideal effects and deviations from two-dimensionality are significantly reduced with respect to other 2D fluid dynamics experiments, such as soap films, electrolyte layers, and rotating tanks. This is due to the fact that, under experimental conditions in which the cold, non-relativistic guiding center approximation is valid, the transverse dynamics of the electron plasma column in the trap is well described by the drift-Poisson equations [1]

\[
\frac{\partial n}{\partial t} + \mathbf{v} \cdot \nabla n = 0 \quad \mathbf{v} = -\nabla \phi \times \hat{e}_z B \quad \nabla^2 \phi = \frac{en}{\varepsilon_0}
\]

where \(n\) is the plasma density, \(\mathbf{v}\) the fluid velocity, \(\phi\) the electrostatic potential, \(B\) the magnetic field, \(\hat{e}_z\) the unit vector in the axial (magnetic field) direction, \(-e\) the electron charge, and \(\varepsilon_0\) the vacuum permittivity. Eqs. (1) are isomorphic to the 2D Euler equations for an incompressible, inviscid fluid

\[
\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla \zeta = 0 \quad \mathbf{v} = -\nabla \psi \times \hat{e}_z \quad \nabla^2 \psi = \zeta
\]
with vorticity $\zeta = en/\varepsilon_0$ and stream function $\psi = \phi/B$.

The behavior of freely decaying 2D turbulence in pure electron plasmas has been extensively studied by using variational principles (see e.g. Refs. [2, 3, 4]) or by analyzing the time evolution of the number of coherent vortices [5]. In recent years, the statistics and dynamics of 2D turbulence in an electron plasma has been studied with both Fourier transforms [6] and wavelet analysis [7, 8]. In this work the free-decaying 2D turbulence in a pure electron plasma confined in the Malmberg-Penning trap ELTRAP is investigated experimentally and analyzed through the Proper Orthogonal Decomposition (POD). POD is used to extract coherent structures of the flow that are energetically dominant and to study their spatio-temporal dynamics over the onset and relaxation phases of turbulence.

**PROPER ORTHOGONAL DECOMPOSITION**

The POD (see e.g. [9]) was developed in 40’s and first introduced in the context of turbulence by Lumley [10]. The POD is a powerful technique for extracting from an ensemble of observations (obtained from experiments or numerical simulations) a basis of empirical functions (eigenfunctions) that identify energetic coherent structures present in a turbulent flow. Given an ensemble of fields, i.e. in most cases a set of observations of a field $u(r, t)$ (for example velocity or vorticity) measured at different times $t$, the goal is to obtain an orthogonal set of basis functions $\varphi_j(r)$ so that every member of the ensemble can be expanded as

$$u(r, t) = \sum_{j=1}^{\infty} a_j(t) \varphi_j(r) ,$$

(3)

where $a_j(t)$ are temporal modal coefficients of the $j$-th mode. The most remarkable property of POD is *optimality*, that is, the possibility of capturing, on average, the maximum energy possible for a projection on a given number of modes, with respect to any other projection. A basis element is optimal if the average projection of $u(r, t)$ onto $\varphi_j(r)$ is maximized constrained to the unitary norm $||\varphi_j(r)|| = 1$. This constrained optimization problem can be formulated in terms of the calculus of variations and leads to the eigenvalue equation

$$\int_\Omega R(r, r') \varphi(r') dr' = \lambda \varphi(r) ,$$

(4)

where $\Omega$ denotes the spatial domain of the experimental measures, and the kernel is the average autocorrelation function $R(r, r') = \langle u(r)u(r') \rangle$ (where $\langle \cdot \rangle$ denotes the time average over the experimental time series). The optimal basis elements $\varphi_j(r)$ are thus the eigenfunctions of Eq. (4) and they are usually called empirical eigenfunctions since they are derived directly from experiments. The eigenvalues are ordered so that $\lambda_j \geq \lambda_{j+1}$ and represent the average “energy” contained in each mode $j$ (e.g. kinetic energy if $u(r, t)$ is a velocity field or enstrophy if it is a vorticity field). From the diagonal decomposition of the autocorrelation function it follows that $\langle a_j(t)a_k(t) \rangle = \delta_{jk} \lambda_k$, that is, the temporal coefficients are uncorrelated on average and related to the mean energy.
EXPERIMENTAL RESULTS AND DATA ANALYSIS

In this work the POD technique is used to study the freely decaying 2D turbulence in an electron plasma. The experimental data have been obtained in the Malmberg-Penning trap ELTRAP \cite{11}. The time evolution of the system is investigated through an injection-hold-dump cycle and monitored by means of an optical diagnostic system. After being injected into the device, the electrons are trapped for a given time and then dumped onto a phosphor screen. The light emitted by the screen is collected by a charge-coupled device (CCD) camera, so that the light intensity measured at a given position on the CCD sensor is proportional to the axially averaged electron density. A 2D image acquired by the CCD provides thus the density distribution and represents also the vorticity $\zeta(x,y,t)$ of the 2D fluid. The time evolution is studied by repeating the above described machine cycle several times with fixed injection parameters and increasing the trapping time. The shot-to-shot reproducibility of initial conditions is very high, as the typical variation of the measured charge at a given position is less than 0.1 %.

The time sequences acquired as described above are analyzed through the POD technique. In this case the POD expansion is

$$\zeta(x,y,t) = \sum_{j=1}^{N} a_j(t) \phi_j(x,y),$$

and the eigenvalues $\lambda_j$ represent the mean enstrophy of each mode $j$. The number $N$ of modes found from the application of POD to an experimental dataset is equal to the number of snapshots of the time series. In our case we have $N = 250$ frames with a trapping time step of 2 $\mu$s. When performing POD on a vorticity time series without pre-processing, we find that the first POD mode corresponds to the enstrophy time average and it is characterized by an almost constant modal coefficient. This is a typical result of POD when applied to signals having non zero mean. Since we are interested in investigating the dynamics of the system, we pre-process the time series by subtracting the vorticity time average from each frame. A typical evolution of 2D vorticity is shown in Figure 1. The first frame (corresponding to a trapping time $\tau = 2 \mu$s) reflects the shape of the spiral cathode distorted by the diocotron (or Kelvin-Helmholtz in the fluid case) instability \cite{12}, which rapidly leads to a nonlinear evolution of the flow. Several small vortices form, which then interact with each other through close encounters giving rise to merger events and emission of vorticity filaments.

The spectrum of normalized eigenvalues of POD modes for the vorticity sequence of Figure 1 is shown in Figure 2. The normalized eigenvalues represent the fraction of mean enstrophy contained in each POD mode. The kink at $j = 12$ suggests the presence of different dynamical regimes described by the modes with $j \leq 12$ and $j > 12$ respectively. The empirical eigenfunctions and the time evolution of the modal coefficients $a_j(t)$ for the modes $j = 1, 2, 7, 9$ are reported in Figures 3 and 4 respectively. The other modes with $j \leq 12$ have similar structures and time evolutions. The eigenfunctions of these modes are characterized by the presence of coherent structures with size of the order of 5-6 mm in agreement with the results of Bettega et al. \cite{8} based on wavelet analysis. The modal coefficients show oscillations dominated by few discrete frequency components between $\simeq 14$ kHz and $\simeq 84$ kHz with a separation of $\simeq 14$ kHz between nearby
FIGURE 1. Evolution of the plasma density for the analyzed sequence. The trapping time is indicated at the top left corner of each frame.

FIGURE 2. Spectrum of POD mode eigenvalues for the sequence of Figure 1.

components. For instance, the modes $j = 1, 2$ show a main oscillation at $\simeq 56$ kHz, the $j = 7$ mode at $\simeq 28$ kHz, and the $j = 9$ mode at $\simeq 84$ kHz. The fact that the oscillation frequencies are equally spaced suggests that the dominant POD modes can be identified.

FIGURE 3. Empirical eigenfunctions of the modes $j = 1, 2, 7, 9$ obtained from the POD of the sequence in Figure 1.
FIGURE 4. Time evolution of the modal coefficients $a_j(t)$ of the modes $j = 1, 2, 7, 9$ obtained from the POD of the sequence in Figure 1.

with diocotron modes [12]. A preliminary analysis based on linear theory of diocotron modes for an initial, constant density ring indicates that the main diocotron modes are the $l = 4, 5, 6$ modes, in agreement with the spatial structure of the first eigenfunctions. The time behavior of the modal coefficients indicates that these modes are active over the whole evolution of the plasma, that is, both during the initial onset phase and the subsequent relaxation of turbulence.

CONCLUSIONS

The results of this work show that the POD technique applied to 2D turbulent flows in a pure electron plasma allows to identify the dynamical processes and the associated coherent structures which give the dominant contributions to the evolution of the system. A time sequence of 2D electron density (fluid vorticity) measurements, obtained starting from an annular vorticity distribution, was analyzed. The structure of the eigenfunctions of the dominant modes and the evolution of the corresponding temporal coefficients, which show oscillations with few equally spaced frequencies, indicate that the dominant POD modes can be identified with diocotron modes, as confirmed by a preliminary analysis based on linear theory. Moreover, the time behavior of the modal coefficients show that these modes are active over the whole evolution of the plasma, that is, during both the onset and relaxation phases of turbulence.
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REFERENCES