Damping of Bernstein–Greene–Kruskal modes in collisional plasmas

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In this paper, the effect of Coulomb collisions on the stability of Bernstein–Greene–Kruskal (BGK) modes [I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. 108, 546 (1957)] is analyzed by comparing the numerical results of collisional particle-in-cell (PIC) simulations with the theoretical predictions by Zakharov and Karpman [V. E. Zakharov and V. I. Karpman, Sov. Phys. JETP 16, 351 (1963)], for the collisional damping of nonlinear plasma waves. In the absence of collisions, BGK modes are undamped nonlinear electrostatic oscillations, solutions of the Vlasov–Poisson equations; in these structures nonlinearity manifests as the formation of a plateau in the resonant region of the particle distribution function, due to trapping of resonant particles, thus preventing linear Landau damping. When particle-particle Coulomb collisions are effective, this plateau is smoothed out since collisions drive the velocity distribution towards the Maxwellian shape, thus destroying the BGK structure. As shown by Zakharov and Karpman in 1963, under certain assumptions, an exponential time decay with constant damping rate is predicted for the electric field amplitude and a linear dependence of the damping rate on the collision frequency is found. In this paper, the theory by Zakharov and Karpman is revisited and the effects of collisions on the stability of BGK modes and on the long time evolution of nonlinear Landau damping are numerically investigated. The numerical results are obtained through a collisional PIC code that reproduces a physical phenomenology also observed in recent experiments with trapped pure electron plasmas. © 2008 American Institute of Physics. [DOI: 10.1063/1.2837519]

I. INTRODUCTION

Electrostatic plasma waves of vanishing amplitude are known to be exponentially damped in time even in the absence of Coulomb collisions between particles. The linear theory of Landau\(^1\) shows that the energy dissipation is due to the resonant interaction between the wave and those particles whose velocities are sufficiently close to the wave phase velocity. For monotonically decreasing distributions of particle velocities, the wave amplitude is damped exponentially in time and, in the case of Maxwellian velocity distributions, the damping rate reads

\[
\gamma_t = \sqrt{\frac{\pi \omega_p}{8 k^3 \lambda_D^3}} \exp\left[-\left(1 + 3k^2\lambda_D^2\right)/(2k^2\lambda_D^2)\right]
\]

which is exponentially small for large wave phase velocities \(v_p\). In Eq. (1), \(\omega_p\) is the electron plasma frequency, \(k\) is the wavenumber, and \(\lambda_D\) is the Debye length (here, we consider the case of high frequency oscillations, then ions are considered as a motionless background of neutralizing positive charge).

In 1965 O’Neil\(^2\) showed that nonlinear effects like particle trapping can turn off Landau damping. The saturation of damping is related to the fact that trapping oscillations of particles in the potential well of the wave flatten the velocity distribution in the region around \(v_p\), creating a plateau in the resonant region. If the time for wave damping \(\tau_d = \gamma_t^{-1}\) is large compared to the trapping time \(\tau_d = 2\pi\sqrt{m/(eEk)}\), where \(m\) and \(e\) are the electron mass and charge and \(E\) is the electric field amplitude, the electric oscillations display a preliminary linear Landau-type decay, then nonlinear effects stop the damping and the wave amplitude starts oscillating around an almost constant saturation level. In the time asymptotic limit, the trapping oscillations in the electric field envelope disappear and the wave goes on propagating at constant amplitude. Many theoretical\(^3\) and numerical\(^4–9\) studies on the so-called nonlinear Landau damping regime of plasma waves substantiated the O’Neil view, according to which trapping oscillations are responsible for damping saturation, through the distortion of the velocity distribution in the resonant region.

These nonlinear plasma waves driven by trapping oscillations can be thought of as Bernstein–Greene–Kruskal (BGK) type waves.\(^10\) The BGK modes are nonlinear stationary solutions of the Vlasov–Poisson equations characterized by a population of trapped particles and whose nonlinearity appears as a distortion of the particle velocity distribution in the region around the wave phase velocity. When collisions are neglected, the main manifestation of this distortion is the formation of a plateau in the trapped region; because of this distortion, the electric oscillations are completely undamped in time.

The goal of the work presented in this paper is to study the effect of Coulomb collisions on the stability of these waves. First, we revisit the theory by Zakharov and Karpman (ZK theory),\(^11\) who in 1963 discussed the solution of the kinetic equations for electrostatic waves of finite but small amplitude, in a weakly collisional plasma. Within ZK theory velocity diffusion is described by the one-dimensional Lenard–Bernstein collisional operator\(^12\) that is of the

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Fokker–Planck form and has the important property of yielding the Maxwell distribution for the equilibrium state. Zakharov and Karpman found an exponential damping of the wave amplitude in the presence of collisions, under the assumption of quasistationary conditions; in simpler words, they obtained an exponential decrement of the wave amplitude provided the collisional damping rate $\gamma_c$ is small (we will discuss in the following the limits of validity of this theory). An interesting consequence of the theoretical ZK model is that the collisional damping rate is dependent on the electrostatic potential as $\gamma_c \propto q^3 v^{3/2}$ and linearly proportional to the collision frequency $v$.

As a second step, we numerically analyze, through particle-in-cell (PIC) simulations, the time evolution of nonlinear plasma waves in both weakly and strongly collisional plasmas. PIC simulations numerically integrate the equations of motion of a large number of electrons (typically each simulation follows the dynamics of $N = 2 \times 10^7$ particles) and Coulomb collisions are explicitly added to the simulations using a Langevin model (the details will be given in the following).

In the first set of simulations, the BGK mode is excited by forcing the plasma with an external driver electric field in the absence of collisions, in such a way to flatten the particle velocity distribution in the vicinity of the wave phase velocity. Then collisions are turned on after the BGK structure has been created through the driving process. Several runs are performed in the range of validity of the ZK theory, in order to substantiate the analytical results. Finally, the time evolution of the electric field oscillations is investigated numerically in the case of strong collisional effects, where the ZK model is no longer valid.

In the second set of simulations, we consider an initial value problem and discuss the evolution of finite amplitude electrostatic oscillations, focusing on the effect of collisions on the long time regime of nonlinear Landau damping. Here collisions are present since the initial time $t=0$, in such a way we numerical reproduce a real experimental situation. Our numerical results show that, for low collision frequency, the effect of collisions appears in the time evolution of the electric field as a slow decay in the wave amplitude, superimposed to the nonlinear trapping oscillations. This numerical evidence is in agreement with the analytical results by Brodin, who in 1997 theoretically investigated the nonlinear Landau damping regime of plasma waves in weakly collisional plasmas, using the Lenard–Bernstein collision model. The slow decay in the oscillations of the electric field envelope has been also observed by Danielson et al. in a recent experiment with standing plasma waves in a trapped pure electron plasma, confined in a Penning–Malmberg apparatus. Through this experiment, the authors showed that a large amplitude wave initially damps in time at the Landau rate, then oscillates, and finally approaches a BGK steady state, that slowly decays due to dissipation; the observed collisional damping rate of the wave amplitude is in agreement with the analytical prediction by Zakharov and Karpman in 1963, obtained in the framework of the Lenard–Bernstein theory. The excited modes in this experiment are discrete oscillations of the plasma column which cannot resonantly couple to sideband oscillations, so that sideband frequency generation cannot prevent the approach to a BGK equilibrium, as for previous experimental works.

We will discuss in the following, our numerical simulations reproduce the experimental situation described in Ref. 16 and are in agreement with the analytical work by Brodin. Finally, we found that when the collision frequency is larger than a critical value $v^*$, no plasma waves can be launched in the regime of nonlinear Landau damping.

The paper is organized as follows: In Sec. II the model of collisional damping proposed by Zakharov and Karpman in 1963 is revisited and the limits of validity of this theoretical model are discussed. Section III is devoted to the numerical simulations of the propagation of nonlinear plasma waves in collisional plasmas. A short summary of our results is given in Sec. IV.

II. THE THEORY OF COLLISIONAL DAMPING

In a collisional plasma, the competition of nonlinear trapping that forms a plateau in the resonant region, and Coulomb collisions, which tend to drive the system towards the Maxwellian equilibrium, determines the final form of the velocity distribution. In other words, if collisions are sufficiently efficient, the plateau formed by nonlinear trapping is smoothed out due to collisions and the wave energy can be damped in time.

The model proposed by Zakharov and Karpman to theoretically describe this phenomenon is based on the kinetic equation for the time evolution of the particle distribution function in a weakly collisional plasma, written in the Fokker–Planck form,

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + e \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial v} = \nu \frac{\partial}{\partial v} \left( \frac{\partial f}{\partial v^2} + vf \right),$$

(2)

where $f$ is the electron distribution function, $\Phi$ is the electrostatic potential, and $\nu$ is the collision frequency. Equation (2) has been obtained in the approximation of high frequency waves, i.e., dropping terms of order $(\bar{v}/v_\phi)^2$, and specializing for electrons whose velocity is close to $v_\phi$. In the ZK model, collisionality is considered low, in the sense that trapped particles bounce many times in the wave trough, before being collisionally detrapped.

The form of the collision operator in Eq. (2) is the same used by Lenard and Bernstein in 1958 to study the linear evolution of plasma oscillations in presence of small-angle collisions. This collision term is a simplified form of the Fokker–Planck collision operator that represents electron-electron and electron-ion encounters. This simplified Lenard–Bernstein operator neglects the velocity dependence of the collision frequency, but preserves important properties of the Fokker–Planck operator; the property of conserving the number of electrons, the property of representing diffusion in velocity space, and finally the property of yielding the Maxwell distribution for the equilibrium state.

Equation (2) gets much simpler to solve if one moves to the rest system of the wave and assumes that the wave damping is small, thus neglecting the time derivative term $\partial f/\partial t$. 

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In order to keep the validity of this stationary approximation, the time variations in the electric potential amplitude must be slow. Within the ZK model, the electric potential $\Phi$ is taken to be a monochromatic wave in the form $\Phi(x, t) = \phi(t)[1 - \cos(kx)]/2$ (in the wave frame), whose initial amplitude $\phi_0 = \phi(t=0)$ is finite but small.

With these assumptions, Eq. (2) can be solved separately for two regions of the particle distribution function, the trapped region and the free region [the trapped region is defined by the condition $|\phi| \leq \Delta v = (2 e v_f / m)^{1/2}$], and finally the equation for the time evolution of the amplitude of the electric potential can be written in the following form:

$$\frac{d\phi(t)}{dt} = -\gamma_c(t)\phi(t),$$  

(3)

$$\gamma_c(t) = \frac{2}{3} \alpha v \phi^{-3/2} \phi_0^{-3/2}, \quad \alpha = \frac{12(7\pi + 6)}{16\sqrt{\pi}} v_0^3 e^{-v_0^2/2}.$$  

(4)

For the sake of simplicity, in the previous and in the next formulas, time is scaled by the inverse electron plasma frequency $\omega_p^2$ and length by the Debye length $\lambda_D$; consequently, velocities are scaled by the electron thermal speed $\bar{v}$. The previous set of equations can be solved to find

$$\phi(t) = \phi_0[1 - \alpha v \phi^{-3/2} \phi_0^{-3/2}]^{2/3}.$$  

(5)

In the following, we discuss the validity of the stationary approximation used to solve Eq. (2). As mentioned before, the stationary approximation is valid as long as the damping of the wave amplitude in the presence of collisions is small; to be more quantitative, the characteristic time for damping $\gamma_c^{-1}$ must remain much larger than the time $\tau_1$ for establishing local equilibrium in the resonant region $\Delta v$ due to collisions. Following the ZK approach, this characteristic time $\tau_1$ can be written in the form $\tau_1 = 3(2\pi)^{-1}(\Delta v / \bar{v}_e)^2$ (see Ref. 11 for details). The above definition of $\tau_1$ together with Eqs. (4) and (5) can be used to write the inequality $\gamma_c \tau_1 \ll 1$ in the equivalent form,

$$t < \frac{\phi_0^{3/2}}{\alpha v} - \frac{8 \alpha^2}{\bar{v}_e^6} = t_{bd},$$  

(6)

where of course it must be

$$\frac{\phi_0^{3/2}}{\alpha v} - \frac{8 \alpha^2}{\bar{v}_e^6} > 0.$$  

(7)

We point out that the width of the trapped region in scaled units is given by $\Delta v = \sqrt{2} \phi_0$. The inequality in Eq. (6) gives the time $t_{bd}$ for the breakdown of the stationary approximation, once fixed the initial amplitude of the electric potential $\phi_0$, the wavenumber $k$, the collision frequency $\nu$, and the wave phase velocity $v_\phi$. By a simple manipulation, from Eq. (6) one obtains

$$\alpha v \phi_0^{-3/2} \phi_0^{-1} \ll 1 - \frac{8 \alpha^3}{\bar{v}_e^6} \phi_0^{1/2} < 1,$$  

(8)

where $8 \alpha^3/(\bar{v}_e^6 \phi_0^{3/2})$ is a positive quantity. It is worth noting that the condition expressed in Eq. (8) is consistent with the fact that the amplitude of the electric potential must decay slowly in time. Moreover, using condition (8) to Taylor expand Eq. (5), one gets

$$\phi(t) = \phi_0[1 - \frac{7}{3} \alpha v \phi^{-3/2} \phi_0^{-3/2}] = \phi_0 e^{-\gamma_c^* t},$$  

(9)

where

$$\gamma_c^* = \frac{7}{3} \alpha v \phi_0^{-3/2}$$  

(10)

is the collisional damping rate (constant in time) obtained within the ZK model.

We conclude that the ZK model describes the exponential damping of nonlinear plasma waves in collisional plasmas, with constant damping rate $\gamma_c^*$. To keep the validity of this model, condition (6) must be satisfied.

### III. PARTICLE-IN-CELL SIMULATIONS

The propagation of plasma waves in unmagnetized plasmas can be described in a 1D-1V phase space (1D in physical space and 1D in velocity space). As we discussed in the Introduction, the PIC simulations follow the dynamics of $N = 2 \times 10^7$ simulation particles, integrating numerically their equations of motion through a standard second order leapfrog scheme, under the effect of the electrostatic field $E(x, t)$. Particles move in the $x$ direction, that is the direction of wave propagation; periodic boundary conditions are imposed in physical space, then Poisson’s equation for the electric field is solved using a standard Fast Fourier Transform routine.

For convenience, in the simulations, time is scaled by the inverse plasma frequency $\omega_p^{-1}$, length is scaled by the Debye length $\lambda_D$. With these choices, velocity is scaled by the electron thermal speed $\bar{v} = \lambda_D \omega_p$ and the electric field by $\sqrt{4\pi e n m_0 \bar{v}^2}$.

The numerical phase space domain is $D = [0, L_x] \times [-v_{\text{max}}, v_{\text{max}}]$, where $v_{\text{max}} = 6$ and $L_x = 20$. The time step is $\Delta t = 0.1$ and the maximum time for the simulations is $t_{\text{max}} = 1600$. The initial distribution of particle velocities is taken to be a Maxwellian.

For a PIC simulation with one spatial dimension and with grid spacing smaller than the Debye length $\Delta x / \lambda_D < 1/6$ in our simulations), the numerical collision time is longer than $t = \alpha v^{-1} n \lambda_D$, where $n = N / L_x$ is the 1-D density of particles in the simulation. In scaled units, the collision time is larger than $t = N / L_x = 10^6$ (for $L_x = 20 \lambda_D$ and $N = 2 \times 10^7$), which is much longer than duration of our runs $t_{\text{max}} = 1600$.

To determine the effect of collisions on the stability of BGK modes, we explicitly add collisions to the simulation using a Langevin model. The equation of motion for each electron is taken to be

$$\frac{dv}{dt} = E(x, t) + R(i) - \nu v,$$  

(11)

where $-\nu v$ is the collisional drag, $\nu$ is the collision frequency, and $R(i)$ is the stochastic acceleration. At each time step, $\Delta t$, the stochastic acceleration term provides a velocity step $\delta v$ of random sign. This step produces velocity diffusion with diffusion coefficient $D_\nu = (\delta v)^2 / 2 \Delta t$. The diffusion coefficient is related to the collision frequency through the
adiabatically. The driver amplitude is near fundamental wave number. The driver is turned on and off responds to the Langmuir frequency trapping period.* 

\[ E_{k} \] 

been turned off. During the driving process, the electric field [Dashed line in the plot indicates the time when the driver has turned on collisions and observe the collisional damping of the BGK mode, previously created through the driving process.

The external driver electric field is taken to be of the form

\[ E_{D}(x,t) = E_{D}^{\text{max}} \left[ 1 + \left( \frac{t - \tau}{\Delta \tau} \right)^{P} \right]^{-1} \sin(mkx - \omega_{D}t), \]  \hspace{1cm} (12)

where \( \tau = 150, \Delta \tau = 100, p = 16, \) and \( k = 2\pi/L_{x} = \pi/10 \) is the fundamental wave number. The driver is turned on and off adiabatically. The driver amplitude is near \( E_{D}^{\text{max}} \) for about a trapping period \( (\tau_{T} = 2\pi/\sqrt{kE_{D}^{\text{max}}}) \) and is near zero again by \( t_{\text{off}} = 300. \) Collisions are turned on for \( t > 500. \) In all the simulations discussed in the present paper, the mode with the largest wavelength that fits in the simulation domain \( (m = 1) \) is excited during the driving process.

Figure 1 shows the time evolution of the plasma electric field spectral component, \( E_{k}(t) \), in the absence of collisions, for a driver frequency \( \omega_{D} = 1.138 \) (the value \( \omega_{D} = 1.138 \) corresponds to the Langmuir frequency \( \omega_{L} = \sqrt{1 + 3k^{2}} \)) and for a maximum driver amplitude \( E_{D}^{\text{max}} = 0.0024. \) The vertical dashed line in the plot indicates the time when the driver has been turned off. During the driving process, the electric field \( E_{k}(t) \) grows to large amplitude, then oscillates around a constant value \( E_{BGK} = 0.051 \) \( (E_{BGK} = 2E_{k}^{\text{sat}} \) where \( E_{k}^{\text{sat}} \) is the saturation amplitude of the electric field spectral component displayed in Fig. 1) and rings at this amplitude after the driver is turned off.

In order to prove that the electric oscillation obtained through the driving process actually is a BGK structure, in the top plot of Fig. 2 we show the phase space contour plot of the resonant electron distribution function at \( t = 500 \) (right before collisions are turned on). The typical vortex structure, signature of the trapping of particles in the wave potential well, is clearly visible in the region around the wave phase velocity \( v_{p} = 3.692 \) for \( k = \pi/10 \), which has been accurately evaluated through a careful Fourier analysis. The velocity width of the trapped particle region (the region of close lines in the contour plot) is \( \Delta v^{\text{num}} = 1.17, \) which is in agreement with the theoretical expectation \( \Delta v^{\text{th}} = 1.17, \) calculated with the amplitude of the electric field at \( t = 500, E_{500} = 0.0535. \)

In order to compare the value of the phase velocity of the undamped mode produced through the driving process with the analytical results of the kinetic theory of waves in plasmas, we numerically solved the electrostatic Vlasov dispersion relation for a BGK-type velocity distribution (with a plateau of width \( \Delta v^{\text{num}} \) in velocity space) such as, for example, \( f(v) = f_{M}(v) - [f_{M}(v) - f_{M}(v_{p})][1 + 2(\nu - v_{p})/\Delta v^{\text{num}}]^{-1} \) [where \( f_{M}(v) \) is a Maxwellian function]. This calculation yields the phase velocity \( v_{BGK}^{\text{num}} = 3.692 \) for \( k = \pi/10, \) in agreement with our numerical results. The same

FIG. 1. Time evolution of the electric field spectral component \( E_{k}(t). \) Excitation of the BGK mode through the driving process in a collisionless plasma.

FIG. 2. (Color online) (Top) Phase space level lines of the resonant particle distribution function. (Bottom) Scatter plot of the distribution \( f \) as a function of the energy in the wave frame \( w. \)
calculation carried with a plateau of vanishing width in velocity \((\Delta v \to 0)\) and consequently \(f \to f_M\) returns the value \(v_{BGK} = 3.735\), increased with respect to the case of finite width plateau. This consideration shows that the wave phase velocity of BGK modes is effectively shifted downward, owing to the finite width of the nonlinear plateau in the velocity distribution. This is a well-known effect that has been discussed in previous studies on BGK waves.\(^{21,22}\)

As an additional check, at the bottom in Fig. 2 we plot the distribution function \(f\) as a function of the energy in the wave frame \(w = (v - v_{\phi})^2 / 2 - \Phi(x, t = 500)\), at \(t = 500\). In other words, for each \((x, v)\) in the numerical domain, we plot \(f(x, v)\) as a function of \(w(x, v)\), resulting in the single curve at the bottom in Fig. 2. The fact that all the points in this scatter plot lie on a single curve shows that the distribution function \(f\) depends on the energy \(w\) alone, as expected for a BGK distribution. Then we conclude that the undamped electrostatic wave excited by the external driver actually is a BGK mode.

For \(t > 500\) collisions are turned on in the simulation. In order to compare our numerical results with the theoretical prediction of the ZK model, we numerically analyze the evolution of electrostatic oscillations in a weakly collisional plasma, with a collision frequency \(\nu = 2 \times 10^{-5}\). Using this value for \(\nu\), the value \(E_{BGK} = 0.051\) for the amplitude of the BGK mode, and its phase velocity \(v_{\phi} = 3.6875\), to calculate the time \(t_{bd}\) for the break down of the ZK model from Eq. (6), one gets \(t_{bd} = 2840\). In our simulation, we follow the system evolution for a time interval \(\Delta t = 1100\) after collisions are turned on, then our numerical results are in the range of validity of the ZK model.

In 2006, Valenti et al.\(^ {21}\) numerically investigated the effect of collisions on the time evolution of a different kind of BGK modes, the so-called electron acoustic waves (EAWs). These authors showed that an EAW of large amplitude \(E_{EAW} = 0.1\), with a wavenumber \(k = \pi / 10\), propagates with a phase velocity \(v_{\phi} = 1.7\). It is straightforward to prove that condition (7) is not satisfied with these EAW parameters; in particular, the ZK model is not valid for oscillations (such as the EAWs) whose phase velocity is close to the electron thermal speed.

In the top plot of Fig. 3, we show the time evolution of the BGK mode, previously produced in the absence of collisions (see Fig. 1), for \(t > 500\), i.e., in the collisional regime. The time when collisions are turned on in the simulation is indicated in the figure by a vertical dashed line. As it is clearly visible, the electric field amplitude is damped slowly in time, as expected for such a small value of the collision frequency \((\nu = 2 \times 10^{-5})\). The bottom panel in the same figure shows a semilogarithmic plot of the time evolution of the envelope of the electric field spectral component \(E_k(t)\). The dashed line in the figure represents the theoretical expectation for the damping rate \(\gamma_c\) in Eq. (10), evaluated with \(v_{\phi} = 3.6875\) and \(E_{BGK} = 0.051\). The figure shows very good agreement between analytical and numerical results.

Repeating such simulation for several values of the collision frequency \(\nu\) (keeping fixed the other parameters) yields the graph in Fig. 4, where the dependence of the collisional damping rate \(\gamma_c\) obtained in the simulations on the collision frequency \(\nu\) is shown. As expected from the ZK theory, a linear dependence is clearly displayed in the figure; the dashed line indicates the ZK theoretical prediction in Eq. (10), calculated using the value \(v_{\phi} = 3.6875\) for the wave phase velocity and \(E_{BGK} = 0.051\) for the amplitude of the electric field.

As we discussed in the Introduction, the role of Coulomb collisions is to smooth out the plateau formed in the BGK distribution function and drive the plasma towards the Max-
In order to show the effect of collisions in phase space, we doubled the amplitude of the excited BGK mode, by varying the amplitude of the external driver, thus increasing the velocity width of the trapped region; by doing so, the phase space plots result clearer than in the case of small amplitude BGK oscillations. Since increasing the BGK amplitude produces a decreasing of the collisional damping rate, according to Eq. (10), in this new simulation we also increase the value of the collision frequency to $\nu = 4.5 \times 10^{-5}$, in such a way that collisional effects are still visible in the electric field evolution. The resulting BGK mode is excited with amplitude $E_{BGK} = 0.094$ and phase velocity $v_\phi = 3.6625$; in this case, the time for the breakdown of the ZK model is $t_{bd} = 3450$ and the theoretical ZK damping rate for this BGK mode is $\gamma^*_c = 0.000164$ (of the same order of magnitude of the damping rate obtained for $E_{BGK} = 0.051$, with $\nu = 2 \times 10^{-5}$).

In Fig. 5, we report the phase-space trajectories of $N_T = 25,000$ particles in the trapped region around the wave phase velocity at different times. Each point in these scatter plots represents the position $x_j$ and the velocity $v_j$ of the $j$th particle at time $t$. For $t \leq 500$, that is, in the absence of collisions, particle trajectories appear as closed paths and well defined structures are visible in phase space. These velocity stripes formed in the trapped region get thinner and thinner in time, before collisions are turned on ($400 \leq t \leq 500$). This phase space vortex propagate in the positive $x$ direction, as it is clear from the first three panels in the figure. For $t > 500$ collisions are turned on in the simulation and they start playing a crucial role in the time evolution of the system; their effect is strongly visible in phase space, even though the value of the collision frequency is very small. In fact, at $t = 550$ the particle trajectories start diffusing in velocity and at larger times ($t = 1000$ and $t = 1600$) the phase space structures, created during the driving process, totally disappear due to a collisional phase mixing; in particular, the separatrix between trapped region and free region is no more visible in the long time limit, meaning that particles, that were trapped in the wave trough, are detrapped by the effect of diffusion in velocity.

A new simulation has been performed in a situation of stronger collisionality. The time evolution of the electric field spectral component $E_x(t)$ is reported in Fig. 6, for $\nu = 5 \times 10^{-4}$ (more than one order of magnitude larger than in the previous run). The time for the breakdown of the ZK theory, corresponding to $\nu = 5 \times 10^{-4}$, $v_\phi = 3.6625$ and $E_{BGK} = 0.094$, is $t_{bd} = 310$. As it is visible from the top plot in the figure, the strong effect of collisions makes the electric oscillations vanish for $t > 1200$. In the bottom panel of the same figure, we show the semilogarithmic plot of the electric field envelope. This plot clearly shows that the wave amplitude damping is not exponential and the theoretical prediction of the ZK model (indicated by a dashed line in the plot) is not valid, except for a short time interval $\delta t < t_{bd}$, after collisions are turned on; in fact, the electric field is rapidly dissipated and reaches the level of the numerical noise ($10^{-9}$) around $t = 1200$.

The effect of collisions on the BGK distribution function is displayed in Fig. 7, for $\nu = 5 \times 10^{-4}$. In the six panels of Fig. 7, the distribution function $f(x_M, v)$ is reported in a semilogarithmic plot as a function of velocity $v$, for a given
point in physical space $x_M$, at different times. The point $x_M$ corresponds to the coordinate in physical space where the velocity width of the trapped region is maximum. The first panel corresponds to the time $t=500$, right before collisions are turned on; a large nonlinear distortion is visible in the region around $v=v_0\approx 3.6625$. It is worth nothing that in the absence of collisions, for finite wave amplitudes the nonlinear distortion of the velocity distribution at the wave phase velocity, caused by trapping of resonant particles, looks more like a hole than a plateau, as it is clearly visible in the first plot of Fig. 7. This effect has been previously observed, for example, in Ref. 4, for lower amplitude nonlinear plasma waves. As it is easily seen from Fig. 7, increasing time collisions smooth out this distortion and finally drive the distribution towards the Maxwellian shape, represented in the plots by the dashed line, thus completely damping the electric oscillations.

**B. Initial value problem**

In this section, we focus on the production of nonlinear plasma waves (BGK-type oscillations) through the so-called nonlinear Landau damping phenomenon. As we discussed in the Introduction, in the absence of collisions electrostatic waves are damped exponentially in time as predicted by Landau in 1946, except in the case when their amplitude is large enough that nonlinearities get effective, before the wave energy is totally dissipated by linear Landau damping. In this case, Landau damping is stopped due to the nonlinear distortion of the velocity distribution in the resonant region, caused by trapping of resonant particles in the potential well of the wave. In other words, nonlinear wave-particle interaction turns off Landau damping, by producing a BGK-type velocity distribution.

When the plasma is not fully collisionless, the possibility of exciting such nonlinear waves, as the result of nonlinear energy exchange between wave and trapped particles, depends on the competition between trapping oscillations, that try to make the distribution flat at the wave phase velocity, and collisions, that tends to maintain the Maxwellian velocity distribution.

In this new set of simulations, the initial distribution function is a Maxwellian in velocity space, over which a perturbation in physical space with amplitude $A=0.064$ and wave number $k=\pi/10$ is superposed,

$$f(x,v,t=0) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}[1 + A \cos(kx)].$$  \hspace{1cm} (13)

In all simulations, the mode with the largest wavelength that fits in the numerical domain is excited at $t=0$, thus allowing us to prevent sideband frequency generation in the system. In the simulations presented in this section, collisions are present since the initial time $t=0$. The value of the trapping frequency corresponding to the perturbation amplitude $A=0.064$ is $\omega_T=2\pi/\tau_T=\sqrt{A}=0.25$, while the linear Landau damping rate, corresponding to $k=\pi/10$, is $\gamma_L=0.028$; consequently, the ratio between the trapping frequency and the linear damping rate is $\omega_T/\gamma_L=10$, meaning that we are considering the trapping regime, where linear Landau damping is stopped by nonlinear effects, as predicted by O’Neil in 1965. \textsuperscript{5}

The time evolution of the perturbation imposed at $t=0$ is followed up to a time $t=600$. In Fig. 8, the time evolution of the electric field spectral component $E_L(t)$ is shown in the absence of collisions. After the initial Landau damping stage, nonlinear effects stop the wave dissipation and the electric amplitude goes on oscillating around a constant saturation level. These nonlinear plasma oscillations can be considered BGK-type structures, in the sense that they are sustained against Landau damping by trapping oscillations.

To quantitatively analyze the competition of trapping oscillations and Coulomb collisions, we introduce the ratio be-
tween the time for collisional velocity diffusion to smooth out a plateau of width $\Delta v$ in the velocity distribution $[\tau_{\text{diff}}=(\Delta v)^2/2D_v=(\Delta v)^2/2\nu]$ and the trapping time $(\tau_T=2\pi/\sqrt{E_0k})$, that is the characteristic time for nonlinear effects to stop the wave damping. For a plateau width $\Delta v = \sqrt{2E_0/k}$, we get $\tau_{\text{diff}}=E_0/(k\nu)$ and finally, in scaled units, the parameter $r$ can be written as

$$r = \frac{\tau_{\text{diff}}}{\tau_T} = \frac{E_0^{3/2}}{2\pi
u k^{1/2}}, \quad (14)$$

where $E_0=E(t=0)=A/k$ is the initial amplitude of the electric field.

In 2006, Valentini et al.21 studied the effects of Coulomb collisions in terms of the parameter $r$, in the excitation of nonlinear EAWs through an external forcing process; they concluded that the condition $r>1$ specifies a threshold for the launching of EAWs.

Here, we analyze the effect of collisions on the possibility of obtaining stable nonlinear plasma waves, as a result of Landau damping saturation due to particle trapping. Then, we use Eq. (13) as an initial condition and numerically follow the evolution of the electrostatic perturbation imposed to the system at $t=0$ in a weakly collisional plasma. For low collisionality, particle trapping is still able to flatten the velocity distribution thus sustaining the electric oscillations, but we find a threshold value for the collision frequency beyond which no stable plasma waves survive in the simulations.

In Fig. 9, the time evolution of the electric perturbation is shown for three different values of the parameter $r$, i.e., for three different values of the collision frequency $\nu$. In the top panel, a low collisionality situation is displayed ($r=564.13$, corresponding to $\nu=4.6 \times 10^3$). In this case, the nonlinear wave propagates after the saturation of the linear damping, but the effect of collisions produces a slow decay in the trapping oscillations of the electric field envelope (compare the top panel of Fig. 9 with the collisionless case shown in Fig. 8). In 2007 Valentini and Veltri23 proposed a physical explanation for this slow decay, that has also been observed in a recent experiment with pure electron plasmas (see Ref. 16). These authors showed that the oscillations in the wave envelope decay due to a collisional mixing, as soon as the velocity displacement due to collisional diffusion gets of the order of the velocity width of the stripes in phase space (see Fig. 5); in other words, the oscillations disappear when the phase space structures, formed in the absence of collisions owing to particle trapping, are no longer resolved in phase space.

When decreasing the value of $r$, the wave damping gets stronger and stronger; in the middle panel of Fig. 9, for $r=50.77$ (corresponding to $\nu=5.1 \times 10^{-4}$), the first two trapping oscillations are visible in the time evolution of the electric field, but for $t>200$ the wave amplitude is quickly damped by collisional effects. Finally, in the bottom panel, for $r=5.64$ (corresponding to $\nu=4.6 \times 10^{-3}$), the plasma wave disappears for $t>100$. The simulations discussed above are in agreement with the analytical results obtained by Brodin in 1997,15 who showed that, in a weakly collisional plasma, in a regime of nonlinear Landau damping there is a tendency for the wave amplitude to have a slow decay superimposed on the nonlinear trapping oscillations. The decay predicted by Brodin is slightly visible in the top plot of Fig. 9, and it is accentuated in the middle plot of the same figure, for a large value of the collision frequency. We conclude that $r \gg 1$ is a necessary condition for the production of nonlinear plasma waves, as a result of saturation of linear Landau damping due to particle trapping. In other words, once fixed the initial amplitude of the perturbation $E_0$ and its wavenumber $k$, plasma waves in a regime of nonlinear Landau damping can be launched in a collisional plasma only if $\nu \ll \nu^* = E_0^{3/2} / (2\pi k^{1/2})$. Using the parameters of our initial value problem simulations, the value of the threshold collision frequency is $\nu^* \approx 0.0254$.

As a last consideration, we compare the time evolution of the electric perturbation for three values of $r$ close to 1. This comparison is reported in the semilogarithmic plots of Fig. 10. At the top, the electric oscillations are shown for $r=5.64$ (corresponding to $\nu=4.6 \times 10^{-3}$), in the middle for $r=1.41$ (corresponding to $\nu=0.019$), while at the bottom for $r=0.507$ (corresponding to $\nu=0.05$). It is clear from the figure that in the three cases the damping of the wave amplitude is exponential in time. The dashed lines in the three plots represent the theoretical (collisionless) linear Landau
damping rate $\gamma_L$ in Eq. (1). As it is easily seen from the figure, for $r=5.64$, particle trapping faintly tries to sustain the oscillations against Landau damping; a weak trapping oscillation is visible around $t=40$, but it is quickly destroyed by collisions. For $t<40$, the electric field decay is exponential with damping rate $\gamma_L$. In the middle panel, no trapping oscillations (even weak) are visible and the wave damping rate is still close to $\gamma_L$. Decreasing more and more the parameter $r$ to values less than unity (i.e., $\nu > \nu^*$), collisions prevent the nonlinear distortion in the velocity distribution, that even remains Maxwellian; in this case, the damping rate of the wave dissipation is enhanced with respect to $\gamma_L$, as it is visible in the bottom plot of Fig. 10.

To point out the dependence of the wave damping on the collision frequency, in the regime of strong collisionality, we repeated the initial condition simulations for values of the collision frequency beyond the critical threshold $\nu^*$ (i.e., $r<1$). As it is clear from Fig. 11, a linear dependence of the damping rate on the collision frequency is observed in the range $\nu > \nu^*$. The numerical dots are fitted by the function $q+\nu p$, indicated in the figure by a dashed line; the parameters of this fit are $q=0.024,$ $p=0.6659$. To complete the analysis, we analyzed the dependence of the damping rate on the initial perturbation amplitude in the case of strong collisionality ($\nu > \nu^*$). We fixed the value of the collision frequency $\nu=0.05$ (corresponding to $r=0.522$) and repeated the simulation for different values of the initial perturbation amplitude in the range $0.01 \lesssim A \lesssim 0.07$. No changes have been observed in the damping rate varying the perturbation amplitude. Therefore, we conclude that in the regime of strong collisionality, collisions prevent nonlinear effects to take place in the system evolution and a frictional damping of the wave is recovered. These results are in agreement with the theoretical predictions in Ref. 12.

In conclusion, our numerical results show the existence of a critical value $\nu^*$ of the collision frequency, that determines the time evolution of nonlinear plasma waves in presence of Coulomb collisions; in particular, for $\nu < \nu^*$, trapping oscillations turn off linear Landau damping and BGK-type waves are generated, in the so-called nonlinear Landau damping regime; for $\nu \approx \nu^*$, the competition between trapping oscillations and particle collisions prevent the generation of nonlinear plasma waves and the electric field amplitude is damped exponentially in time, with damping rate close to $\gamma_L$. Finally, when $\nu > \nu^*$, the system is fully dominated by collisions and the wave amplitude is exponentially damped in time; in this case, the damping rate enhanced with respect to the linear Landau damping rate, linearly proportional to the collision frequency $\nu$, as shown in Fig. 11, and completely independent on the initial amplitude of the electrostatic perturbation (frictional damping).

IV. SUMMARY

In this paper we numerically investigated the evolution of nonlinear plasma waves and BGK modes in the presence of particle velocity diffusion. The BGK modes are undamped oscillations in a collisionless regime, since they are sustained against Landau damping by nonlinear trapping oscillations, that flatten the particle velocity distribution in the region around the wave phase velocity. Collisions work to drive the
velocity distribution towards the Maxwellian shape, thus destroying the BGK structure and dissipating the wave energy.

In order to numerically analyze the effect of collisions on nonlinear oscillations, we used an external forcing electric field to excite the BGK mode; collisions are then turned on to damp the wave. For low collisionality, the wave damping produced by collisions is exponential in time with constant damping rate, as predicted by Zakharov and Karpman in 1963; we reproduced this behavior in the PIC simulations. On the other hand, for high values of the collision frequency, the wave energy is dissipated faster, with a time dependent damping rate.

We also investigated the collisional effect on the production of nonlinear plasma waves, as the result of the nonlinear Landau damping phenomenon, this being a real plasma situation recently analyzed in important experiments with non-neutral plasmas. A critical value for the collision frequency establishes a threshold for producing plasma waves. In fact, while in absence of collisions particle trapping turns off linear Landau damping, by generating a BGK-type velocity distribution, when collision frequency is beyond a threshold value, the system is fully dominated by collisions and the BGK distribution cannot be produced. This numerical result can be of relevant importance for experimental situations, where the collisionless approximation cannot be completely satisfied.

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